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On explicit order 1.5 approximations with varying coefficients: the case of super-linear diffusion coefficients

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Abstract

A conjecture appears in [Kumar and Sabanis (2016). `arXiv:1601.02695[math.PR]`], in the form of a remark, where it is stated that it is possible to construct, in a specified way, any high order explicit numerical schemes to approximate the solutions of SDEs with superlinear coefficients. We answer this conjecture to the positive for the case of order 1.5 approximations and show that the suggested methodology works. Moreover, we explore the case of having Hölder continuous derivatives for the diffusion coefficients.
AMS subject classifications: Primary 60H35; secondary 65C30.

1 Introduction

Due to recent research (see [9], [7], [14], [15], [4] and references therein), new explicit Euler-type schemes have been developed to approximate SDEs with superlinearly growing coefficients following the observation in [8] that the classical (explicit) Euler scheme cannot be used for such approximations. This has been extended to Milstein-type schemes (see [11], [1] and references therein). Such schemes are explicit and therefore more computationally efficient compared to the implicit methods.

In this article, a new type of explicit order 1.5 scheme is constructed. The techniques used in [15] and [11] are further extended to obtain the $\mathcal{L}^2$ rate of convergence of the proposed order 1.5 scheme. The main idea is to follow the approach of [13] by using an appropriate Ito-Taylor (known also as Wagner-Platen) expansion and the taming technique introduced in [15] and [11]. Theorem 1 below gives the rate of convergence in $\mathcal{L}^2$ which is obtained under certain conditions (also given below). In addition, by the combination of the results in [15], [11] and in this article, one can arguably anticipate that, by using the uniform taming approach as explained below, any high order (explicit) scheme can be constructed with the desired rate of convergence as in the global Lipschitz case (see [13]).

Recent developments in data science attracted our attention to the fact that high order schemes can be used for MCMC algorithms with improved convergence properties in high dimensions. Moreover, such schemes can be combined with multilevel techniques in a natural way One can refer to the article [2] on tamed unadjusted Langevin algorithms and consider possible extensions of such techniques to achieve higher accuracy when using tamed schemes to

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sample from a target distribution, typically the invariant measure of the associated Langevin SDE.

This section is concluded by introducing some notation. The Euclidean norm of a vector $b \in \mathbb{R}^d$ and the Hilbert-Schmidt norm of a matrix $\sigma \in \mathbb{R}^{d\times m}$ are denoted by $|b|$ and $|\sigma|$ respectively. σ^* is the transpose matrix of σ. The i-th element of b and (i,j)-th element of σ are denoted respectively by $b^{(i)}$ and $\sigma^{(i,j)}$, for every $i = 1,\dots,d$ and $j = 1,\dots,m$. In addition, $\lfloor a \rfloor$ denotes the integer part of a positive real number a. The inner product of two vectors $x, y \in \mathbb{R}^d$ is denoted by xy. Let $f : \mathbb{R}^d \to \mathbb{R}$ be a twice continuously differentiable function. Denote by ∇f and $\nabla^2 f$ the gradient and the Hessian of f respectively. For every $j = 1,\dots,m$, define $L^0 : C^2(\mathbb{R}^d) \to C(\mathbb{R}^d)$ and $L^j : C^2(\mathbb{R}^d) \to C^1(\mathbb{R}^d)$ by

$$L^0 = \sum_{u=1}^{d} b^{(u)} \frac{\partial}{\partial x^{(u)}} + \frac{1}{2} \sum_{u,l=1}^{d} \sum_{j_1=1}^{m} \sigma^{(u,j_1)} \sigma^{(l,j_1)} \frac{\partial^2}{\partial x^{(u)} \partial x^{(l)}}, \quad L^j = \sum_{u=1}^{d} \sigma^{(u,j)} \frac{\partial}{\partial x^{(u)}}.$$

Note that, for any $j = 1,\dots,m$, by composing the operator L^j with itself, one obtains $L^j L^{j_1} : C^2(\mathbb{R}^d) \to C(\mathbb{R}^d)$ for every $j, j_1 = 1,\dots,m$, which can be written as

$$L^j L^{j_1} = \sum_{u,l=1}^{d} \sigma^{(u,j)} \frac{\partial}{\partial x^{(u)}} \sigma^{(l,j_1)} \frac{\partial}{\partial x^{(l)}} + \sum_{u,l=1}^{d} \sigma^{(u,j)} \sigma^{(l,j_1)} \frac{\partial^2}{\partial x^{(u)} \partial x^{(l)}}.$$

2 Main results

Let $(\Omega, \{\mathscr{F}_t\}_{t\geq 0}, \mathscr{F}, \mathbb{P})$ be a complete filtered probability space satisfying the usual conditions, which means that the filtration is right continuous and $\mathscr{F}_0$ contains all $\mathbb{P}$-null sets. Denote by $(w_t)_{t\in[0,T]}$ an m-dimensional Wiener process. Moreover, assume that b and σ are Borel-measurable functions from $\mathbb{R}^d$ to $\mathbb{R}^d$ and $\mathbb{R}^{d\times m}$, respectively. The drift and diffusion coefficients b and σ are assumed to be twice continuously differentiable in $x \in \mathbb{R}^d$. For a fixed $T > 0$, consider a d-dimensional SDE,

$$x_t = x_0 + \int_0^t b(x_s)\, ds + \int_0^t \sigma(x_s)\, dw_s, \tag{21}$$

almost surely for any $t \in [0,T]$, where x_0 is an $\mathcal{F}_0$-measurable random variable. Let $p_0 \geq 4$, $p_1 > 2$, and $\rho \geq 2$. The following assumptions are stated.

A-1 $\mathbb{E}|x_0|^{p_0} < \infty$.

A-2 There exists a constant $K > 0$, such that for any $x \in \mathbb{R}^d$,

$$2xb(x) + (p_0 - 1)|\sigma(x)|^2 \leq K(1 + |x|^2).$$

A-3 There exists a constant $K > 0$, such that for any $x, \bar{x} \in \mathbb{R}^d$,

$$2(x - \bar{x})(b(x) - b(\bar{x})) + (p_1 - 1)|\sigma(x) - \sigma(\bar{x})|^2 \leq K|x - \bar{x}|^2.$$

A-4 There exists a constant $K > 0$, such that for any $x, \bar{x} \in \mathbb{R}^d$, and $i = 1,\dots,d$,

$$|\nabla^2 b^{(i)}(x) - \nabla^2 b^{(i)}(\bar{x})| \leq K(1 + |x| + |\bar{x}|)^{\rho-2}|x - \bar{x}|.$$

A-5 There exist constants $K > 0$ and $\beta \in (0,1]$, such that for any $x, \bar{x} \in \mathbb{R}^d$, $i = 1,\dots,d$, and $j = 1,\dots,m$,

$$|\nabla^2 \sigma^{(i,j)}(x) - \nabla^2 \sigma^{(i,j)}(\bar{x})| \leq K(1 + |x| + |\bar{x}|)^{\frac{\rho-4}{2}}|x - \bar{x}|^{\beta}.$$

Remark 1. *Assume **A-4** and **A-5** hold. Then, one can obtain the following estimates in a straightforward manner. In particular, by **A-4**, there exists a constant K>0, such that for any $i,u,l=1,\dots,d$, and $x,\bar{x}\in\mathbb{R}^d$,*

$$\left|\frac{\partial^2 b^{(i)}(x)}{\partial y^{(u)}\partial y^{(l)}}\right| \le K(1+|x|)^{\rho-1}.$$

In addition,

$$\left|\frac{\partial b^{(i)}(x)}{\partial y^{(u)}} - \frac{\partial b^{(i)}(\bar{x})}{\partial y^{(u)}}\right| \le K(1+|x|+|\bar{x}|)^{\rho-1}|x-\bar{x}|.$$

Furthermore, there is a constant $K>0$ such that for any $i,u=1,\dots,d$, and $x,\bar{x}\in\mathbb{R}^d$,

$$\left|\frac{\partial b^{(i)}(x)}{\partial y^{(u)}}\right| \le K(1+|x|)^{\rho},$$

$$|b(x)-b(\bar{x})| \le K(1+|x|+|\bar{x}|)^{\rho}|x-\bar{x}|,$$

which implies

$$|b(x)| \le K(1+|x|)^{\rho+1}.$$

*Similarly, by **A-5**, there exists $K>0$, such that for any $i,u,l=1,\dots,d$, $j=1,\dots,m$ and $x\in\mathbb{R}^d$,*

$$\left|\frac{\partial^2 \sigma^{(i,j)}(x)}{\partial y^{(u)}\partial y^{(l)}}\right| \le K(1+|x|)^{\frac{\rho-2}{2}},$$

Moreover, there exists $K>0$, such that for any $j=1,\dots,m$ and $x,\bar{x}\in\mathbb{R}^d$,

$$\left|\frac{\partial \sigma^{(i,j)}(x)}{\partial y^{(u)}} - \frac{\partial \sigma^{(i,j)}(\bar{x})}{\partial y^{(u)}}\right| \le K(1+|x|+|\bar{x}|)^{\frac{\rho-2}{2}}|x-\bar{x}|.$$

Furthermore, there exists K>0, such that for any $i,u=1,\dots,d$, $j=1,\dots,m$ and $x,\bar{x}\in\mathbb{R}^d$,

$$\left|\frac{\partial \sigma^{(i,j)}(x)}{\partial y^{(u)}}\right| \le K(1+|x|)^{\frac{\rho}{2}},$$

$$|\sigma(x)-\sigma(\bar{x})| \le K(1+|x|+|\bar{x}|)^{\frac{\rho}{2}}|x-\bar{x}|,$$

which implies

$$|\sigma(x)| \le K(1+|x|)^{\frac{\rho}{2}+1}.$$

Then, there exists a constant $K>0$, such that

$$|L^0 b(x)| \le K(1+|x|)^{2\rho+1}, \quad |L^j b(x)| \le K(1+|x|)^{\frac{3}{2}\rho+1},$$

$$|L^0 \sigma(x)| \le K(1+|x|)^{\frac{3}{2}\rho+1}, \quad |L^j \sigma(x)| \le K(1+|x|)^{\rho+1},$$

$$|L^j L^{j_1} \sigma(x)| \le K(1+|x|)^{\frac{3}{2}\rho+1}.$$

We adopt a uniform taming approach meaning that all terms of interest in the numerical scheme, which are used to approximate the SDE (21), are controlled in the same way, i.e. $\frac{1}{1+n^{-\theta}|x|^{2\rho\theta}}$ is used where θ represents the desired rate. More concretely, in the order 1.5 paradigm, one constructs, for any $n\in\mathbb{N}$ and $f\in C^2(\mathbb{R}^d)$,

$$f^n(x) = \frac{f(x)}{1+n^{-\theta}|x|^{2\rho\theta}}, \quad L^{n,0}f(x) := \frac{L^0 f(x)}{1+n^{-\theta}|x|^{2\rho\theta}},$$

$$L^{n,j}f(x) := \frac{L^j f(x)}{1+n^{-\theta}|x|^{2\rho\theta}}, \quad L^{n,j}L^{j_1}f(x) := \frac{L^j L^{j_1} f(x)}{1+n^{-\theta}|x|^{2\rho\theta}},$$

where θ is taken to be $3/2$.

Remark 2. *Throughout this article, the constant $C>0$ may take different values at different places, but it is always independent of $n \in \mathbb{N}$.*

Remark 3. *Due to Remark 1, one observes that, there exists a constant $C>0$, such that for any $n \in \mathbb{N}$*

$$|b^n(x)| \leq \min\left(Cn^{\frac{1}{2}}(1+|x|), |b(x)|\right), \quad |\sigma^n(x)|^2 \leq \min\left(Cn^{\frac{1}{2}}(1+|x|^2), |\sigma(x)|^2\right),$$

$$|L^{n,0}b(x)| \leq \min\left(Cn(1+|x|), |L^0 b(x)|\right), \quad |L^{n,j}b(x)| \leq \min\left(Cn^{\frac{3}{4}}(1+|x|), |L^j b(x)|\right),$$

$$|L^{n,0}\sigma(x)| \leq \min\left(Cn^{\frac{3}{4}}(1+|x|), |L^0 \sigma(x)|\right), \quad |L^{n,j}\sigma(x)| \leq \min\left(Cn^{\frac{1}{2}}(1+|x|), |L^j \sigma(x)|\right),$$

$$|L^{n,j}L^{j_1}\sigma(x)| \leq \min\left(Cn^{\frac{3}{4}}(1+|x|), |L^j L^{j_1}\sigma(x)|\right).$$

Define $\kappa(n,t) := \lfloor nt \rfloor / n$, for any $t \in [0,T]$. Denote by

$$b_1^n(t,x) = \int_{\kappa(n,t)}^t L^{n,0}b(x)\,ds, \quad b_2^n(t,x) = \sum_j \int_{\kappa(n,t)}^t L^{n,j}b(x)\,dw_s^j,$$

$$\tilde{b}^n(t,x) = b^n(x) + b_1^n(t,x) + b_2^n(t,x),$$

$$\sigma_1^n(t,x) = \sum_j \int_{\kappa(n,t)}^t L^{n,j}\sigma(x)\,dw_s^j, \quad \sigma_2^n(t,x) = \int_{\kappa(n,t)}^t L^{n,0}\sigma(x)\,ds,$$

$$\sigma_3^n(t,x) = \sum_j \sum_{j_1} \int_{\kappa(n,t)}^t \int_{\kappa(n,t)}^s L^{n,j}L^{j_1}\sigma(x)\,dw_r^j\,dw_s^{j_1},$$

$$\tilde{\sigma}^n(t,x) = \sigma^n(x) + \sigma_M^n(t,x),$$

where $\sigma_M^n(t,x) = \sigma_1^n(t,x) + \sigma_2^n(t,x) + \sigma_3^n(t,x)$. The order 1.5 strong Taylor scheme is as follows:

$$x_t^n = x_0 + \int_0^t \tilde{b}^n(s, x_{\kappa(n,s)}^n)\,ds + \int_0^t \tilde{\sigma}^n(s, x_{\kappa(n,s)}^n)\,dw_s, \tag{22}$$

almost surely for any $t \in [0,T]$.

Theorem 1. *Assume **A-1** - **A-5** are satisfied with $p_0 \geq 2(5\rho+1)$, then the explicit order 1.5 scheme (22) converges to the true solution of the SDE (21) in $\mathcal{L}^2$ with a rate of convergence equal to $1+\beta/2$, i.e., there exists a constant $C>0$, such that for any $n \in \mathbb{N}$,*

$$\left(\sup_{0\leq t\leq T} \mathbb{E}|x_t - x_t^n|^2\right)^{1/2} \leq Cn^{-(1+\beta/2)}. \tag{23}$$

3 Moment bounds

Lemma 1. *Assume **A-1** - **A-3** hold. Then, there is a unique solution to the SDE* (21)*, and the p_0-th moment of the solution is bounded uniformly in time, i.e. there exists a constant $C > 0$, such that for any $t \in [0, T]$,*

$$\sup_{0 \le t \le T} \mathbb{E}|x_t|^{p_0} \le C.$$

Proof. It is a well-known result, and the proof can be found in [12]. □

Remark 4. *By Remark 3, for each $n \in \mathbb{N}$, the norm of $\tilde{b}^n$ and $\tilde{\sigma}^n$ are growing at most linearly in x. Then, together with **A-1**, this guarantees that for any $n \in \mathbb{N}$ and $p \le p_0$,*

$$\mathbb{E}\left[\sup_{0 \le t \le T} |x_t^n|^p\right] < \infty.$$

Lemma 2. *Let **A-4** - **A-5** be satisfied, then there exists a constant $C > 0$, such that for any $n \in \mathbb{N}$ and $t \in [0, T]$,*

$$\mathbb{E}|b_1^n(t, x_{\kappa(n,t)}^n)|^{p_0} \le C(1 + \mathbb{E}|x_{\kappa(n,t)}^n|^{p_0}),$$
$$\mathbb{E}|b_2^n(t, x_{\kappa(n,t)}^n)|^{p_0} \le Cn^{\frac{p_0}{4}}(1 + \mathbb{E}|x_{\kappa(n,t)}^n|^{p_0}),$$
$$\mathbb{E}|\sigma_1^n(t, x_{\kappa(n,t)}^n)|^{p_0} \le C(1 + \mathbb{E}|x_{\kappa(n,t)}^n|^{p_0}),$$
$$\mathbb{E}|\sigma_2^n(t, x_{\kappa(n,t)}^n)|^{p_0} \le C(1 + \mathbb{E}|x_{\kappa(n,t)}^n|^{p_0}),$$
$$\mathbb{E}|\sigma_3^n(t, x_{\kappa(n,t)}^n)|^{p_0} \le C(1 + \mathbb{E}|x_{\kappa(n,t)}^n|^{p_0}).$$

Proof. Due to Remark 3, these inequalities follow immediately. □

Corollary 1. *Assume **A-4** - **A-5** are satisfied, then there exists a constant $C > 0$, such that for any $n \in \mathbb{N}$ and $t \in [0, T]$,*

$$\mathbb{E}|\tilde{b}^n(t, x_{\kappa(n,t)}^n)|^{p_0} \le Cn^{\frac{p_0}{2}}(1 + \mathbb{E}|x_{\kappa(n,t)}^n|^{p_0}),$$

$$\mathbb{E}|\tilde{\sigma}^n(t, x_{\kappa(n,t)}^n)|^{p_0} \le Cn^{\frac{p_0}{4}}(1 + \mathbb{E}|x_{\kappa(n,t)}^n|^{p_0}).$$

Lemma 3. *Assume **A-1** - **A-5** hold, then there exists a constant $C > 0$, such that for any $n \in \mathbb{N}$, the order* 1.5 *scheme* (22) *satisfies*

$$\sup_{n \in \mathbb{N}} \sup_{0 \le t \le T} \mathbb{E}|x_t^n|^{p_0} \le C.$$

Proof. Itô's formula gives, almost surely,

$$\begin{aligned}
|x_t^n|^{p_0} = |x_0|^{p_0} &+ p_0 \int_0^t |x_s^n|^{p_0-2} x_s^n \tilde{b}^n(s, x_{\kappa(n,s)}^n)\, ds \\
&+ p_0 \int_0^t |x_s^n|^{p_0-2} x_s^n \tilde{\sigma}^n(s, x_{\kappa(n,s)}^n)\, dw_s \\
&+ \frac{p_0}{2} \int_0^t |x_s^n|^{p_0-2} |\tilde{\sigma}^n(s, x_{\kappa(n,s)}^n)|^2\, ds \\
&+ \frac{p_0(p_0-2)}{2} \int_0^t |x_s^n|^{p_0-4} |\tilde{\sigma}^{n*}(s, x_{\kappa(n,s)}^n) x_s^n|^2\, ds,
\end{aligned}$$

for any $t \in [0,T]$. Then, since the expectation of the third term above is zero, one obtains

$$
\begin{aligned}
\mathbb{E}|x_t^n|^{p_0} \leq & \mathbb{E}|x_0|^{p_0} + p_0\mathbb{E}\int_0^t |x_s^n|^{p_0-2}(x_s^n - x_{\kappa(n,s)}^n)b^n(x_{\kappa(n,s)}^n)\, ds \\
& + p_0\mathbb{E}\int_0^t |x_s^n|^{p_0-2}x_{\kappa(n,s)}^n b^n(x_{\kappa(n,s)}^n)\, ds \\
& + p_0\mathbb{E}\int_0^t |x_s^n|^{p_0-2}x_s^n b_1^n(s, x_{\kappa(n,s)}^n)\, ds + p_0\mathbb{E}\int_0^t |x_s^n|^{p_0-2}x_s^n b_2^n(s, x_{\kappa(n,s)}^n)\, ds \\
& + \frac{p_0(p_0-1)}{2}\mathbb{E}\int_0^t |x_s^n|^{p_0-2}|\tilde{\sigma}^n(s, x_{\kappa(n,s)}^n)|^2\, ds,
\end{aligned}
$$

which can be written as

$$
\mathbb{E}|x_t^n|^{p_0} \leq G_1 + \sum_{i=2}^{7} G_i(t), \tag{31}
$$

where $G_1 = \mathbb{E}|x_0|^{p_0}$,

$$
\begin{aligned}
G_2(t) &= p_0\mathbb{E}\int_0^t |x_s^n|^{p_0-2}(x_s^n - x_{\kappa(n,s)}^n)b^n(x_{\kappa(n,s)}^n)\, ds, \\
G_3(t) &= \frac{p_0}{2}\mathbb{E}\int_0^t |x_s^n|^{p_0-2}(2x_{\kappa(n,s)}^n b^n(x_{\kappa(n,s)}^n) + (p_0-1)|\sigma^n(x_{\kappa(n,s)}^n)|^2)\, ds, \\
G_4(t) &= p_0\mathbb{E}\int_0^t |x_s^n|^{p_0-2}x_s^n b_1^n(s, x_{\kappa(n,s)}^n)\, ds, \\
G_5(t) &= p_0\mathbb{E}\int_0^t |x_s^n|^{p_0-2}x_s^n b_2^n(s, x_{\kappa(n,s)}^n)\, ds, \\
G_6(t) &= \frac{p_0(p_0-1)}{2}\mathbb{E}\int_0^t |x_s^n|^{p_0-2}|\sigma_M^n(s, x_{\kappa(n,s)}^n)|^2\, ds, \\
G_7(t) &= p_0(p_0-1)\mathbb{E}\int_0^t |x_s^n|^{p_0-2}\sum_{k=1}^{d}\sum_{v=1}^{m}\sigma^{n,(k,v)}(x_{\kappa(n,s)}^n)\sigma_M^{n,(k,v)}(s, x_{\kappa(n,s)}^n)\, ds.
\end{aligned}
$$

In order to estimate $G_2(t)$, one writes

$$
\begin{aligned}
G_2(t) = & p_0\mathbb{E}\int_0^t |x_s^n|^{p_0-2}\int_{\kappa(n,s)}^{s}\tilde{b}^n(r, x_{\kappa(n,r)}^n)\, dr b^n(x_{\kappa(n,s)}^n)\, ds \\
& + p_0\mathbb{E}\int_0^t |x_s^n|^{p_0-2}\int_{\kappa(n,s)}^{s}\tilde{\sigma}^n(r, x_{\kappa(n,r)}^n)\, dw_r b^n(x_{\kappa(n,s)}^n)\, ds,
\end{aligned}
$$

for any $t \in [0,T]$. By applying Young's inequality and Remark 3, the following estimate can be obtained

$$
\begin{aligned}
G_2(t) \leq & C + C\int_0^t \sup_{0\leq r\leq s}\mathbb{E}|x_r^n|^{p_0} ds + C\mathbb{E}\int_0^t \left| n^{\frac{1}{2}}\int_{\kappa(n,s)}^{s}\tilde{b}^n(r, x_{\kappa(n,r)}^n)\, dr\right|^{p_0} ds \\
& + p_0\mathbb{E}\int_0^t (|x_s^n|^{p_0-2} - |x_{\kappa(n,s)}^n|^{p_0-2}|)\int_{\kappa(n,s)}^{s}\tilde{\sigma}^n(r, x_{\kappa(n,r)}^n)\, dw_r b^n(x_{\kappa(n,s)}^n)\, ds \\
& + p_0\mathbb{E}\int_0^t |x_{\kappa(n,s)}^n|^{p_0-2}\int_{\kappa(n,s)}^{s}\tilde{\sigma}^n(r, x_{\kappa(n,r)}^n)\, dw_r b^n(x_{\kappa(n,s)}^n)\, ds,
\end{aligned}
$$

for any $t \in [0,T]$. Since the last term above is zero, by taking into consideration the results of Corollary 1 and by applying Itô's formula, it follows that, almost surely

$$
\begin{aligned}
G_2(t) \leq{}& C + C\int_0^t \sup_{0\leq r\leq s} \mathbb{E}|x_r^n|^{p_0} ds \\
&+ C\mathbb{E}\int_0^t \int_{\kappa(n,s)}^s |x_r^n|^{p_0-4} x_r^n \tilde{b}^n(r, x^n_{\kappa(n,r)})\,dr \int_{\kappa(n,s)}^s \tilde{\sigma}^n(r, x^n_{\kappa(n,r)})\,dw_r b^n(x^n_{\kappa(n,s)})\,ds \\
&+ C\mathbb{E}\int_0^t \int_{\kappa(n,s)}^s |x_r^n|^{p_0-4} x_r^n \tilde{\sigma}^n(r, x^n_{\kappa(n,r)})\,dw_r \int_{\kappa(n,s)}^s \tilde{\sigma}^n(r, x^n_{\kappa(n,r)})\,dw_r b^n(x^n_{\kappa(n,s)})\,ds \\
&+ C\mathbb{E}\int_0^t \int_{\kappa(n,s)}^s |x_r^n|^{p_0-4} |\tilde{\sigma}^n(r, x^n_{\kappa(n,r)})|^2\,dr \Big|\int_{\kappa(n,s)}^s \tilde{\sigma}^n(r, x^n_{\kappa(n,r)})\,dw_r\Big| |b^n(x^n_{\kappa(n,s)})|\,ds.
\end{aligned}
$$

Due to Remark 3,

$$
\begin{aligned}
G_2(t) \leq{}& C + C\int_0^t \sup_{0\leq r\leq s} \mathbb{E}|x_r^n|^{p_0} ds \\
&+ Cn^{\frac{1}{2}}\mathbb{E}\int_0^t \int_{\kappa(n,s)}^s |x_r^n|^{p_0-3}(1+|x^n_{\kappa(n,s)}|)|\tilde{b}^n(r, x^n_{\kappa(n,r)})|\,dr \left|\int_{\kappa(n,s)}^s \tilde{\sigma}^n(r, x^n_{\kappa(n,r)})\,dw_r\right| ds \\
&+ Cn^{\frac{1}{2}}\mathbb{E}\int_0^t \int_{\kappa(n,s)}^s |x_r^n|^{p_0-3}(1+|x^n_{\kappa(n,s)}|)|\tilde{\sigma}^n(r, x^n_{\kappa(n,r)})|^2\,dr\,ds \\
&+ Cn^{\frac{1}{2}}\mathbb{E}\int_0^t \int_{\kappa(n,s)}^s |x_r^n|^{p_0-4}(1+|x^n_{\kappa(n,s)}|)|\tilde{\sigma}^n(r, x^n_{\kappa(n,r)})|^2\,dr \left|\int_{\kappa(n,s)}^s \tilde{\sigma}^n(r, x^n_{\kappa(n,r)})\,dw_r\right| ds,
\end{aligned}
$$

for any $t \in [0,T]$. Then, the application of Young's inequality yields

$$
\begin{aligned}
G_2(t) \leq{}& C + C\int_0^t \sup_{0\leq r\leq s} \mathbb{E}|x_r^n|^{p_0} ds \\
&+ C\mathbb{E}\int_0^t n^{\frac{1}{4}} \int_{\kappa(n,s)}^s (1+|x_r^n|^{p_0-2}+|x^n_{\kappa(n,s)}|^{p_0-2})|\tilde{b}^n(r, x^n_{\kappa(n,r)})|\,dr \\
&\qquad \times n^{\frac{1}{4}}\left|\int_{\kappa(n,s)}^s \tilde{\sigma}^n(r, x^n_{\kappa(n,r)})\,dw_r\right| ds \\
&+ C\mathbb{E}\int_0^t \int_{\kappa(n,s)}^s n^{1-\frac{2}{p_0}}(1+|x_r^n|^{p_0-2}+|x^n_{\kappa(n,s)}|^{p_0-2})n^{-\frac{1}{2}+\frac{2}{p_0}}|\tilde{\sigma}^n(r, x^n_{\kappa(n,r)})|^2\,dr\,ds \\
&+ C\mathbb{E}\int_0^t n^{\frac{1}{4}} \int_{\kappa(n,s)}^s (1+|x_r^n|^{p_0-3}+|x^n_{\kappa(n,s)}|^{p_0-3})|\tilde{\sigma}^n(r, x^n_{\kappa(n,r)})|^2\,dr \\
&\qquad \times n^{\frac{1}{4}}\left|\int_{\kappa(n,s)}^s \tilde{\sigma}^n(r, x^n_{\kappa(n,r)})\,dw_r\right| ds,
\end{aligned}
$$

which can be further estimated as

$$
\begin{aligned}
G_2(t) \leq{}& C + C\int_0^t \sup_{0\leq r\leq s} \mathbb{E}|x_r^n|^{p_0} ds \\
&+ C\mathbb{E}\int_0^t \left(\int_{\kappa(n,s)}^s n^{\frac{3}{4}-\frac{1}{p_0}}(1+|x_r^n|^{p_0-2}+|x^n_{\kappa(n,s)}|^{p_0-2})n^{-\frac{1}{2}+\frac{1}{p_0}}|\tilde{b}^n(r, x^n_{\kappa(n,r)})|\,dr\right)^{\frac{p_0}{p_0-1}} ds \\
&+ Cn\mathbb{E}\int_0^t \int_{\kappa(n,s)}^s (1+|x_r^n|^{p_0}+|x^n_{\kappa(n,s)}|^{p_0})\,dr\,ds
\end{aligned}
$$

$$+ Cn^{-\frac{p_0}{4}+1}\mathbb{E}\int_0^t \int_{\kappa(n,s)}^s |\tilde{\sigma}^n(r, x^n_{\kappa(n,r)})|^{p_0}\, dr\, ds$$

$$+ C\mathbb{E}\int_0^t \left(\int_{\kappa(n,s)}^s n^{\frac{3}{4}-\frac{2}{p_0}}(1+|x^n_r|^{p_0-3}+|x^n_{\kappa(n,s)}|^{p_0-3})n^{-\frac{1}{2}+\frac{2}{p_0}}|\tilde{\sigma}^n(r, x^n_{\kappa(n,r)})|^2\, dr\right)^{\frac{p_0}{p_0-1}} ds$$

$$+ Cn^{\frac{p_0}{4}}\int_0^t \mathbb{E}\left|\int_{\kappa(n,s)}^s \tilde{\sigma}^n(r, x^n_{\kappa(n,r)})\, dw_r\right|^{p_0} ds$$

for any $t \in [0, T]$. By using Young's inequality and Corollary 1, one obtains

$$G_2(t) \le C + C\int_0^t \sup_{0\le r\le s} \mathbb{E}|x^n_r|^{p_0} ds$$

$$+ C\mathbb{E}\int_0^t \left(\int_{\kappa(n,s)}^s n^{\frac{3p_0-4}{4p_0}\times\frac{p_0-1}{p_0-2}}(1+|x^n_r|^{p_0-1}+|x^n_{\kappa(n,s)}|^{p_0-1})\, dr\right)^{\frac{p_0}{p_0-1}} ds$$

$$+ C\mathbb{E}\int_0^t \left(\int_{\kappa(n,s)}^s n^{\frac{(2-p_0)\times(p_0-1)}{2p_0}}|\tilde{b}^n(r, x^n_{\kappa(n,r)})|^{p_0-1}\, dr\right)^{\frac{p_0}{p_0-1}} ds$$

$$+ C\mathbb{E}\int_0^t \left(\int_{\kappa(n,s)}^s n^{\frac{3p_0-8}{4p_0}\times\frac{p_0-1}{p_0-3}}(1+|x^n_r|^{p_0-1}+|x^n_{\kappa(n,s)}|^{p_0-1})\, dr\right)^{\frac{p_0}{p_0-1}} ds$$

$$+ C\mathbb{E}\int_0^t \left(\int_{\kappa(n,s)}^s n^{\frac{4-p_0}{2p_0}\times\frac{p_0-1}{2}}|\tilde{\sigma}^n(r, x^n_{\kappa(n,r)})|^{p_0-1}\, dr\right)^{\frac{p_0}{p_0-1}} ds$$

$$+ Cn^{-\frac{p_0}{4}+1}\int_0^t \int_{\kappa(n,s)}^s \mathbb{E}|\tilde{\sigma}^n(r, x^n_{\kappa(n,r)})|^{p_0}\, dr\, ds,$$

which, due to Hölder's inequality and Corollary 1, implies

$$G_2(t) \le C + C\int_0^t \sup_{0\le r\le s} \mathbb{E}|x^n_r|^{p_0} ds$$

$$+ Cn^{\frac{3p_0-4}{4(p_0-2)}-\frac{1}{p_0-1}}\int_0^t \mathbb{E}\int_{\kappa(n,s)}^s (1+|x^n_r|^{p_0}+|x^n_{\kappa(n,s)}|^{p_0})\, dr\, ds$$

$$+ Cn^{-\frac{p_0}{2}+1-\frac{1}{p_0-1}}\int_0^t \mathbb{E}\int_{\kappa(n,s)}^s |\tilde{b}^n(r, x^n_{\kappa(n,r)})|^{p_0}\, dr\, ds$$

$$+ Cn^{\frac{3p_0-8}{4(p_0-3)}-\frac{1}{p_0-1}}\int_0^t \mathbb{E}\int_{\kappa(n,s)}^s (1+|x^n_r|^{p_0}+|x^n_{\kappa(n,s)}|^{p_0})\, dr\, ds,$$

for any $t \in [0, T]$. Note that in the third and fifth term above, $n^{\frac{3p_0-4}{4(p_0-2)}}$ and $n^{\frac{3p_0-8}{4(p_0-3)}}$ are less than n for all $p_0 \ge 4$. Thus, in view of Corollary 1, one obtains

$$G_2(t) \le C + C\int_0^t \sup_{0\le r\le s} \mathbb{E}|x^n_r|^{p_0} ds,$$

for any $t \in [0, T]$. For $G_3(t)$, applying **A-2** gives

$$G_3(t) = \frac{p_0}{2}\mathbb{E}\int_0^t |x^n_s|^{p_0-2}\frac{2x^n_{\kappa(n,s)}b(x^n_{\kappa(n,s)}) + (p_0-1)|\sigma(x^n_{\kappa(n,s)})|^2}{1+n^{-3/2}|x^n_{\kappa(n,s)}|^{3\rho}} ds$$

$$\le C\mathbb{E}\int_0^t |x^n_s|^{p_0-2}(1+|x^n_{\kappa(n,s)}|^2)\, ds,$$

which, due to Young's inequality, results in

$$G_3(t) \leq C + C\int_0^t \sup_{0\leq r\leq s} \mathbb{E}|x_r^n|^{p_0} ds,$$

for any $t \in [0,T]$. To estimate $G_4(t)$, one uses Young's inequality to obtain

$$G_4(t) \leq C\mathbb{E}\int_0^t |x_s^n|^{p_0}\, ds + C\mathbb{E}\int_0^t |b_1^n(s, x_{\kappa(n,s)}^n)|^{p_0}\, ds,$$

which implies due to Lemma 2,

$$G_4(t) \leq C + C\int_0^t \sup_{0\leq r\leq s} \mathbb{E}|x_r^n|^{p_0} ds,$$

for any $t \in [0,T]$. Moreover, one writes

$$G_5(t) = \sum_{i=1}^{3} G_{5i}(t),$$

where

$$\begin{aligned}
G_{51}(t) &= p_0\mathbb{E}\int_0^t |x_s^n|^{p_0-2}(x_s^n - x_{\kappa(n,s)}^n) b_2^n(s, x_{\kappa(n,s)}^n)\, ds, \\
G_{52}(t) &= p_0\mathbb{E}\int_0^t (|x_s^n|^{p_0-2} - |x_{\kappa(n,s)}^n|^{p_0-2}) x_{\kappa(n,s)}^n b_2^n(s, x_{\kappa(n,s)}^n)\, ds, \\
G_{53}(t) &= p_0\mathbb{E}\int_0^t |x_{\kappa(n,s)}^n|^{p_0-2} x_{\kappa(n,s)}^n b_2^n(s, x_{\kappa(n,s)}^n)\, ds.
\end{aligned}$$

One then calculates the following

$$\begin{aligned}
G_{51}(t) =& p_0\mathbb{E}\int_0^t |x_s^n|^{p_0-2}\int_{\kappa(n,s)}^s \tilde{b}^n(r, x_{\kappa(n,r)}^n)\, dr b_2^n(s, x_{\kappa(n,s)}^n)\, ds \\
&+ p_0\mathbb{E}\int_0^t |x_s^n|^{p_0-2}\int_{\kappa(n,s)}^s \tilde{\sigma}^n(r, x_{\kappa(n,r)}^n)\, dw_r b_2^n(s, x_{\kappa(n,s)}^n)\, ds,
\end{aligned}$$

which implies, due to Young's inequality,

$$\begin{aligned}
G_{51}(t) \leq& C\mathbb{E}\int_0^t |x_s^n|^{p_0}\, ds + C\mathbb{E}\int_0^t \left| n^{\frac{1}{4}}\int_{\kappa(n,s)}^s \tilde{b}^n(r, x_{\kappa(n,r)}^n)\, dr n^{-\frac{1}{4}} b_2^n(s, x_{\kappa(n,s)}^n)\right|^{\frac{p_0}{2}} ds \\
&+ C\mathbb{E}\int_0^t \left| n^{\frac{1}{4}}\int_{\kappa(n,s)}^s \tilde{\sigma}^n(r, x_{\kappa(n,r)}^n)\, dw_r n^{-\frac{1}{4}} b_2^n(s, x_{\kappa(n,s)}^n)\right|^{\frac{p_0}{2}} ds,
\end{aligned}$$

for any $t \in [0,T]$. Then, on applying Young's inequality again, one obtains

$$\begin{aligned}
G_{51}(t) \leq& C\mathbb{E}\int_0^t |x_s^n|^{p_0}\, ds + Cn^{\frac{p_0}{4}}\mathbb{E}\int_0^t \left|\int_{\kappa(n,s)}^s \tilde{b}^n(r, x_{\kappa(n,r)}^n)\, dr\right|^{p_0} ds \\
&+ Cn^{\frac{p_0}{4}}\mathbb{E}\int_0^t \left|\int_{\kappa(n,s)}^s \tilde{\sigma}^n(r, x_{\kappa(n,r)}^n)\, dw_r\right|^{p_0} ds + Cn^{-\frac{p_0}{4}}\mathbb{E}\int_0^t |b_2^n(s, x_{\kappa(n,s)}^n)|^{p_0}\, ds,
\end{aligned}$$

which by using Hölder's inequality and Lemma 2 yields

$$
\begin{aligned}
G_{51}(t) \leq & C + C \int_0^t \sup_{0\leq r\leq s} \mathbb{E}|x_r^n|^{p_0} ds \\
& + C n^{\frac{p_0}{4}-p_0+1} \int_0^t \int_{\kappa(n,s)}^s \mathbb{E}|\tilde{b}^n(r, x_{\kappa(n,r)}^n)|^{p_0}\, dr\, ds \\
& + C n^{\frac{p_0}{4}-\frac{p_0}{2}+1} \int_0^t \int_{\kappa(n,s)}^s \mathbb{E}|\tilde{\sigma}^n(r, x_{\kappa(n,r)}^n)|^{p_0}\, dr\, ds,
\end{aligned}
$$

for any $t \in [0, T]$. Due to Corollary 1, one concludes that

$$
G_{51}(t) \leq C + C \int_0^t \sup_{0\leq r\leq s} \mathbb{E}|x_r^n|^{p_0} ds, \tag{32}
$$

for any $t \in [0, T]$. As for $G_{52}(t)$, Itô's formula gives, almost surely

$$
\begin{aligned}
G_{52}(t) \leq & C\mathbb{E} \int_0^t \int_{\kappa(n,s)}^s |x_r^n|^{p_0-4} x_r^n \tilde{b}^n(r, x_{\kappa(n,r)}^n)\, dr x_{\kappa(n,s)}^n b_2^n(s, x_{\kappa(n,s)}^n)\, ds \\
& + C\mathbb{E} \int_0^t \int_{\kappa(n,s)}^s |x_r^n|^{p_0-4} x_r^n \tilde{\sigma}^n(r, x_{\kappa(n,r)}^n)\, dw_r x_{\kappa(n,s)}^n \sum_j \int_{\kappa(n,s)}^s L^{n,j} b(x_{\kappa(n,r)}^n)\, dw_r^j\, ds \\
& + C\mathbb{E} \int_0^t \int_{\kappa(n,s)}^s |x_r^n|^{p_0-4} |\tilde{\sigma}^n(r, x_{\kappa(n,r)}^n)|^2\, dr |x_{\kappa(n,s)}^n| |b_2^n(s, x_{\kappa(n,s)}^n)|\, ds,
\end{aligned}
$$

which, by Young's inequality, can be expressed as

$$
\begin{aligned}
G_{52}(t) \leq & C \int_0^t \mathbb{E} \int_{\kappa(n,s)}^s n^{\frac{3}{4}-\frac{1}{p_0}} (1 + |x_r^n|^{p_0-2} + |x_{\kappa(n,s)}^n|^{p_0-2}) n^{-\frac{1}{2}+\frac{1}{p_0}} |\tilde{b}^n(r, x_{\kappa(n,r)}^n)|\, dr \\
& \qquad \times n^{-\frac{1}{4}} |b_2^n(s, x_{\kappa(n,s)}^n)|\, ds \\
& + C \sum_{j=1}^m \int_0^t \mathbb{E} \int_{\kappa(n,s)}^s (1 + |x_r^n|^{p_0-2} + |x_{\kappa(n,s)}^n|^{p_0-2}) |\tilde{\sigma}^n(r, x_{\kappa(n,r)}^n)| |L^{n,j} b(x_{\kappa(n,r)}^n)|\, dr\, ds \\
& + C \int_0^t \mathbb{E} \int_{\kappa(n,s)}^s n^{\frac{3}{4}-\frac{2}{p_0}} (1 + |x_r^n|^{p_0-3} + |x_{\kappa(n,s)}^n|^{p_0-3}) n^{-\frac{1}{2}+\frac{2}{p_0}} |\tilde{\sigma}^n(r, x_{\kappa(n,r)}^n)|^2\, dr \\
& \qquad \times n^{-\frac{1}{4}} |b_2^n(s, x_{\kappa(n,s)}^n)|\, ds,
\end{aligned}
$$

for any $t \in [0, T]$. One uses Young's inequality again and Remark 3 to obtain

$$
\begin{aligned}
G_{52}(t) \leq & C \int_0^t \mathbb{E} \left(\int_{\kappa(n,s)}^s n^{\frac{3}{4}-\frac{1}{p_0}} (1 + |x_r^n|^{p_0-2} + |x_{\kappa(n,s)}^n|^{p_0-2}) n^{-\frac{1}{2}+\frac{1}{p_0}} |\tilde{b}^n(r, x_{\kappa(n,r)}^n)|\, dr \right)^{\frac{p_0}{p_0-1}} ds \\
& + C \int_0^t \mathbb{E} \int_{\kappa(n,s)}^s n^{1-\frac{1}{p_0}} (1 + |x_r^n|^{p_0-1} + |x_{\kappa(n,s)}^n|^{p_0-1}) n^{-\frac{1}{4}+\frac{1}{p_0}} |\tilde{\sigma}^n(r, x_{\kappa(n,r)}^n)|\, dr\, ds \\
& + C \int_0^t \mathbb{E} \left(\int_{\kappa(n,s)}^s n^{\frac{3}{4}-\frac{2}{p_0}} (1 + |x_r^n|^{p_0-3} + |x_{\kappa(n,s)}^n|^{p_0-3}) n^{-\frac{1}{2}+\frac{2}{p_0}} |\tilde{\sigma}^n(r, x_{\kappa(n,r)}^n)|^2\, dr \right)^{\frac{p_0}{p_0-1}} ds \\
& + C n^{-\frac{p_0}{4}} \int_0^t \mathbb{E}|b_2^n(s, x_{\kappa(n,s)}^n)|^{p_0}\, ds,
\end{aligned}
$$

which implies due to Lemma 2

$$
G_{52}(t) \leq C \int_0^t \mathbb{E} \left(\int_{\kappa(n,s)}^s n^{\frac{3p_0-4}{4p_0} \times \frac{p_0-1}{p_0-2}} (1 + |x_r^n|^{p_0-1} + |x_{\kappa(n,s)}^n|^{p_0-1})\, dr \right)^{\frac{p_0}{p_0-1}} ds
$$

$$
\begin{aligned}
&+ C\int_0^t \mathbb{E}\left(\int_{\kappa(n,s)}^s n^{\frac{(2-p_0)\times(p_0-1)}{2p_0}}|\tilde{b}^n(r,x^n_{\kappa(n,r)})|^{p_0-1}\,dr\right)^{\frac{p_0}{p_0-1}} ds\\
&+ C\int_0^t \mathbb{E}\int_{\kappa(n,s)}^s n(1+|x^n_r|^{p_0}+|x^n_{\kappa(n,s)}|^{p_0})\,dr\,ds\\
&+ C\int_0^t \mathbb{E}\int_{\kappa(n,s)}^s n^{\frac{4-p_0}{4p_0}\times p_0}|\tilde{\sigma}^n(r,x^n_{\kappa(n,r)})|^{p_0}\,dr\,ds\\
&+ C\int_0^t \mathbb{E}\left(\int_{\kappa(n,s)}^s n^{\frac{3p_0-8}{4p_0}\times\frac{p_0-1}{p_0-3}}(1+|x^n_r|^{p_0-1}+|x^n_{\kappa(n,s)}|^{p_0-1})\,dr\right)^{\frac{p_0}{p_0-1}} ds\\
&+ C\int_0^t \mathbb{E}\left(\int_{\kappa(n,s)}^s n^{\frac{4-p_0}{2p_0}\times\frac{p_0-1}{2}}|\tilde{\sigma}^n(r,x^n_{\kappa(n,r)})|^{p_0-1}\,dr\right)^{\frac{p_0}{p_0-1}} ds\\
&+ C + C\int_0^t \sup_{0\le r\le s}\mathbb{E}|x^n_r|^{p_0}ds,
\end{aligned}
$$

for any $t\in[0,T]$. By using Hölder's inequality and Corollary 1,

$$
\begin{aligned}
G_{52}(t) \le\, & Cn^{\frac{3p_0-4}{4(p_0-2)}-\frac{1}{p_0-1}}\int_0^t \mathbb{E}\int_{\kappa(n,s)}^s (1+|x^n_r|^{p_0}+|x^n_{\kappa(n,s)}|^{p_0})\,dr\,ds\\
&+ Cn^{-\frac{p_0}{2}+1-\frac{1}{p_0-1}}\int_0^t \mathbb{E}\int_{\kappa(n,s)}^s |\tilde{b}^n(r,x^n_{\kappa(n,r)})|^{p_0}\,dr\,ds\\
&+ Cn^{\frac{3p_0-8}{4(p_0-3)}-\frac{1}{p_0-1}}\int_0^t \mathbb{E}\int_{\kappa(n,s)}^s (1+|x^n_r|^{p_0}+|x^n_{\kappa(n,s)}|^{p_0})\,dr\,ds\\
&+ Cn^{-\frac{p_0}{4}+1-\frac{1}{p_0-1}}\int_0^t \mathbb{E}\int_{\kappa(n,s)}^s |\tilde{\sigma}^n(r,x^n_{\kappa(n,r)})|^{p_0}\,dr\,ds\\
&+ C + C\int_0^t \sup_{0\le r\le s}\mathbb{E}|x^n_r|^{p_0}ds,
\end{aligned}
$$

for any $t\in[0,T]$. One observes that $n^{\frac{3p_0-4}{4(p_0-2)}}$ and $n^{\frac{3p_0-8}{4(p_0-3)}}$ are less than n for all $p_0\ge 4$, then due to Corollary 1, the following holds

$$
G_{52}(t) \le C + C\int_0^t \sup_{0\le r\le s}\mathbb{E}|x^n_r|^{p_0}ds, \tag{33}
$$

for any $t\in[0,T]$. In addition, note that by the definition of $b^n_2(t,x)$, one obtains

$$
G_{53}(t) := p_0\mathbb{E}\int_0^t |x^n_{\kappa(n,s)}|^{p_0-2}x^n_{\kappa(n,s)}b^n_2(s,x^n_{\kappa(n,s)})\,ds = 0, \tag{34}
$$

for any $t\in[0,T]$. Then, substituting (32), (33) and (34) into (35), one obtains

$$
G_5(t) \le C + C\int_0^t \sup_{0\le r\le s}\mathbb{E}|x^n_r|^{p_0}ds, \tag{35}
$$

for any $t\in[0,T]$. In order to estimate $G_6(t)$, one applies Young's inequality to obtain

$$
G_6(t) \le C\mathbb{E}\int_0^t |x^n_s|^{p_0}\,ds + C\mathbb{E}\int_0^t |\sigma^n_M(s,x^n_{\kappa(n,s)})|^{p_0}\,ds
$$

$$\leq C\mathbb{E}\int_0^t |x_s^n|^{p_0}\,ds + C\mathbb{E}\int_0^t |\sigma_1^n(s,x_{\kappa(n,s)}^n)|^{p_0}\,ds$$
$$+ C\mathbb{E}\int_0^t |\sigma_2^n(s,x_{\kappa(n,s)}^n)|^{p_0}\,ds + C\mathbb{E}\int_0^t |\sigma_3^n(s,x_{\kappa(n,s)}^n)|^{p_0}\,ds,$$

which implies due to Lemma 2

$$G_6(t) \leq C + C\int_0^t \sup_{0\leq r\leq s} \mathbb{E}|x_r^n|^{p_0} ds,$$

for any $t \in [0,T]$. Finally, for $G_7(t)$, one writes

$$G_7(t) = \sum_{i=1}^{2} G_{7i}(t), \tag{36}$$

where

$$G_{71}(t) = p_0(p_0-1)\mathbb{E}\int_0^t (|x_s^n|^{p_0-2} - |x_{\kappa(n,s)}^n|^{p_0-2}) \sum_{k=1}^{d}\sum_{v=1}^{m} \sigma^{n,(k,v)}(x_{\kappa(n,s)}^n)\sigma_M^{n,(k,v)}(s,x_{\kappa(n,s)}^n)\,ds,$$
$$G_{72}(t) = p_0(p_0-1)\mathbb{E}\int_0^t |x_{\kappa(n,s)}^n|^{p_0-2} \sum_{k=1}^{d}\sum_{v=1}^{m} \sigma^{n,(k,v)}(x_{\kappa(n,s)}^n)\sigma_M^{n,(k,v)}(s,x_{\kappa(n,s)}^n)\,ds.$$

To estimate $G_{71}(t)$, Itô's formula gives, almost surely

$$\begin{aligned}
G_{71}(t) :=& C\mathbb{E}\int_0^t (|x_s^n|^{p_0-2} - |x_{\kappa(n,s)}^n|^{p_0-2}) \sum_{k=1}^{d}\sum_{v=1}^{m} \sigma^{n,(k,v)}(x_{\kappa(n,s)}^n)\sigma_M^{n,(k,v)}(s,x_{\kappa(n,s)}^n)\,ds \\
\leq& C\mathbb{E}\int_0^t \int_{\kappa(n,s)}^s |x_r^n|^{p_0-4} x_r^n \tilde{b}^n(r,x_{\kappa(n,r)}^n)\,dr \sum_{k=1}^{d}\sum_{v=1}^{m} \sigma^{n,(k,v)}(x_{\kappa(n,s)}^n)\sigma_M^{n,(k,v)}(s,x_{\kappa(n,s)}^n)\,ds \\
&+ C\mathbb{E}\int_0^t \int_{\kappa(n,s)}^s |x_r^n|^{p_0-4} x_r^n \tilde{\sigma}^n(r,x_{\kappa(n,r)}^n)\,dw_r \sum_{k=1}^{d}\sum_{v=1}^{m} \sigma^{n,(k,v)}(x_{\kappa(n,s)}^n)\sigma_M^{n,(k,v)}(s,x_{\kappa(n,s)}^n)\,ds \\
&+ C\mathbb{E}\int_0^t \int_{\kappa(n,s)}^s |x_r^n|^{p_0-4} |\tilde{\sigma}^n(r,x_{\kappa(n,r)}^n)|^2\,dr \Big| \sum_{k=1}^{d}\sum_{v=1}^{m} \sigma^{n,(k,v)}(x_{\kappa(n,s)}^n)\sigma_M^{n,(k,v)}(s,x_{\kappa(n,s)}^n)\Big|\,ds,
\end{aligned}$$

which by using Remark 3 implies

$$\begin{aligned}
G_{71}(t) \leq& Cn^{\frac{1}{4}}\mathbb{E}\int_0^t \int_{\kappa(n,s)}^s |x_r^n|^{p_0-3}(1+|x_{\kappa(n,s)}^n|)|\tilde{b}^n(r,x_{\kappa(n,r)}^n)|\,dr|\sigma_M^n(s,x_{\kappa(n,s)}^n)|\,ds \\
&+ C\mathbb{E}\int_0^t \int_{\kappa(n,s)}^s |x_r^n|^{p_0-4} x_r^n \tilde{\sigma}^n(r,x_{\kappa(n,r)}^n)\,dw_r \\
&\quad\times \sum_{k=1}^{d}\sum_{v=1}^{m} \sigma^{n,(k,v)}(x_{\kappa(n,s)}^n) \sum_{j=1}^{m}\int_{\kappa(n,s)}^s L^{n,j}\sigma^{(k,v)}(x_{\kappa(n,r)}^n)\,dw_r^j\,ds \\
&+ C\mathbb{E}\int_0^t \int_{\kappa(n,s)}^s |x_r^n|^{p_0-4} x_r^n \tilde{\sigma}^n(r,x_{\kappa(n,r)}^n)\,dw_r \\
&\quad\times \sum_{k=1}^{d}\sum_{v=1}^{m} \sigma^{n,(k,v)}(x_{\kappa(n,s)}^n) \int_{\kappa(n,s)}^s L^{n,0}\sigma^{(k,v)}(x_{\kappa(n,r)}^n)\,dr\,ds
\end{aligned}$$

$$+ C\mathbb{E}\int_0^t \int_{\kappa(n,s)}^s |x_r^n|^{p_0-4} x_r^n \tilde{\sigma}^n(r, x_{\kappa(n,r)}^n)\, dw_r$$
$$\times \sum_{k=1}^d \sum_{v=1}^m \sigma^{n,(k,v)}(x_{\kappa(n,s)}^n) \sum_{j=1}^m \sum_{j_1=1}^m \int_{\kappa(n,s)}^s \int_{\kappa(n,r)}^r L^{n,j} L^{j_1} \sigma^{(k,v)}(x_{\kappa(n,\gamma)}^n)\, dw_\gamma^{j_1}\, dw_r^j\, ds$$
$$+ Cn^{\frac{1}{4}}\mathbb{E}\int_0^t \int_{\kappa(n,s)}^s |x_r^n|^{p_0-4}(1+|x_{\kappa(n,s)}^n|)|\tilde{\sigma}^n(r, x_{\kappa(n,r)}^n)|^2\, dr |\sigma_M^n(s, x_{\kappa(n,s)}^n)|\, ds,$$

for any $t \in [0, T]$. One then observes that, since $L^{n,0}\sigma(x_{\kappa(n,r)}^n)$ takes the same value for all $r \in [\kappa(n,s), s]$, it can be taken out of the integral in the third term above, and thus the third term is zero. Moreover, by Young's inequality and Remark 3, one obtains

$$G_{71}(t) \leq C\mathbb{E}\int_0^t \int_{\kappa(n,s)}^s n^{\frac{1}{4}}(1+|x_r^n|^{p_0-2}+|x_{\kappa(n,s)}^n|^{p_0-2})|\tilde{b}^n(r, x_{\kappa(n,r)}^n)|\, dr |\sigma_M^n(s, x_{\kappa(n,s)}^n)|\, ds$$
$$+ C\mathbb{E}\int_0^t \int_{\kappa(n,s)}^s n^{\frac{3}{4}}|x_r^n|^{p_0-3}(1+|x_{\kappa(n,s)}^n|)^2|\tilde{\sigma}^n(r, x_{\kappa(n,r)}^n)|\, dr\, ds$$
$$+ C\sum_{j=1}^m \mathbb{E}\int_0^t \int_{\kappa(n,s)}^s n^{\frac{3}{4}-\frac{2}{p_0}}|x_r^n|^{p_0-3}(1+|x_{\kappa(n,s)}^n|)n^{-\frac{1}{4}+\frac{1}{p_0}}|\tilde{\sigma}^n(r, x_{\kappa(n,r)}^n)|$$
$$\times n^{-\frac{1}{4}+\frac{1}{p_0}}\left|\sum_{j_1=1}^d \int_{\kappa(n,r)}^r L^{n,j}L^{j_1}\sigma(x_{\kappa(n,\gamma)}^n)\, dw_\gamma^{j_1}\right| dr\, ds$$
$$+ C\mathbb{E}\int_0^t \int_{\kappa(n,s)}^s n^{\frac{1}{4}}(1+|x_r^n|^{p_0-3}+|x_{\kappa(n,s)}^n|^{p_0-3})|\tilde{\sigma}^n(r, x_{\kappa(n,r)}^n)|^2\, dr |\sigma_M^{n,(i,j)}(s, x_{\kappa(n,s)}^n)|\, ds,$$

which yields, due to Young's inequality,

$$G_{71}(t) \leq C\int_0^t \mathbb{E}\left(\int_{\kappa(n,s)}^s n^{\frac{3}{4}-\frac{1}{p_0}}(1+|x_r^n|^{p_0-2}+|x_{\kappa(n,s)}^n|^{p_0-2})n^{-\frac{1}{2}+\frac{1}{p_0}}|\tilde{b}^n(r, x_{\kappa(n,r)}^n)|\, dr\right)^{\frac{p_0}{p_0-1}} ds$$
$$+ C\mathbb{E}\int_0^t \int_{\kappa(n,s)}^s n^{1-\frac{1}{p_0}}(1+|x_r^n|^{p_0-1}+|x_{\kappa(n,s)}^n|^{p_0-1})n^{-\frac{1}{4}+\frac{1}{p_0}}|\tilde{\sigma}^n(r, x_{\kappa(n,r)}^n)|\, dr\, ds$$
$$+ C\mathbb{E}\int_0^t \int_{\kappa(n,s)}^s \left(n^{\frac{3}{4}-\frac{2}{p_0}}(1+|x_r^n|^{p_0-2}+|x_{\kappa(n,s)}^n|^{p_0-2})n^{-\frac{1}{4}+\frac{1}{p_0}}|\tilde{\sigma}^n(r, x_{\kappa(n,r)}^n)|\right)^{\frac{p_0}{p_0-1}} dr\, ds$$
$$+ C\int_0^t \mathbb{E}\left(\int_{\kappa(n,s)}^s n^{\frac{3}{4}-\frac{2}{p_0}}(1+|x_r^n|^{p_0-3}+|x_{\kappa(n,s)}^n|^{p_0-3})n^{-\frac{1}{2}+\frac{2}{p_0}}|\tilde{\sigma}^n(r, x_{\kappa(n,r)}^n)|^2\, dr\right)^{\frac{p_0}{p_0-1}} ds$$
$$+ C\sum_{j=1}^m n^{-\frac{p_0}{4}+1}\mathbb{E}\int_0^t \int_{\kappa(n,s)}^s \left|\sum_{j_1=1}^d \int_{\kappa(n,r)}^r L^{n,j}L^{j_1}\sigma(x_{\kappa(n,\gamma)}^n)\, dw_\gamma^{j_1}\right|^{p_0} dr\, ds$$
$$+ C\int_0^t \mathbb{E}|\sigma_M^n(s, x_{\kappa(n,s)}^n)|^{p_0}\, ds,$$

for any $t \in [0, T]$. By Young's inequality, Hölder's inequality and Lemma 2,

$$G_{71}(t) \leq Cn^{\frac{3p_0-4}{4(p_0-2)}-\frac{1}{p_0-1}}\int_0^t \mathbb{E}\int_{\kappa(n,s)}^s (1+|x_r^n|^{p_0}+|x_{\kappa(n,s)}^n|^{p_0})\, dr\, ds$$
$$+ Cn^{-\frac{p_0}{2}+1-\frac{1}{p_0-1}}\int_0^t \int_{\kappa(n,s)}^s \mathbb{E}|\tilde{b}^n(r, x_{\kappa(n,r)}^n)|^{p_0}\, dr\, ds$$

$$
\begin{aligned}
&+ Cn\mathbb{E}\int_0^t \int_{\kappa(n,s)}^s (1+|x_r^n|^{p_0}+|x_{\kappa(n,s)}^n|^{p_0})\,dr\,ds \\
&+ Cn^{-\frac{p_0}{4}+1}\int_0^t \int_{\kappa(n,s)}^s \mathbb{E}|\tilde{\sigma}^n(r,x_{\kappa(n,r)}^n)|^{p_0}\,dr\,ds \\
&+ Cn^{\frac{3p_0-8}{4(p_0-2)}}\mathbb{E}\int_0^t \int_{\kappa(n,s)}^s (1+|x_r^n|^{p_0}+|x_{\kappa(n,s)}^n|^{p_0})\,dr\,ds \\
&+ Cn^{-\frac{p_0}{4}+1}\int_0^t \int_{\kappa(n,s)}^s \mathbb{E}|\tilde{\sigma}^n(r,x_{\kappa(n,r)}^n)|^{p_0}\,dr\,ds \\
&+ Cn^{\frac{3p_0-8}{4(p_0-3)}-\frac{1}{p_0-1}}\int_0^t \mathbb{E}\int_{\kappa(n,s)}^s (1+|x_r^n|^{p_0}+|x_{\kappa(n,s)}^n|^{p_0})\,dr\,ds \\
&+ Cn^{-\frac{p_0}{4}+1-\frac{1}{p_0-1}}\int_0^t \int_{\kappa(n,s)}^s \mathbb{E}|\tilde{\sigma}^n(r,x_{\kappa(n,r)}^n)|^{p_0}\,dr\,ds \\
&+ Cn^{-\frac{3p_0}{4}+2}\int_0^t \int_{\kappa(n,s)}^s \int_{\kappa(n,r)}^r \mathbb{E}|L^{n,j}L^{j_1}\sigma(x_{\kappa(n,\gamma)}^n)|^{p_0}\,d\gamma\,dr\,ds \\
&+ C + C\int_0^t \sup_{0\le r\le s}\mathbb{E}|x_r^n|^{p_0}ds,
\end{aligned}
$$

for any $t \in [0,T]$. Due to Corollary 1 and Remark 3, it can be shown that

$$
G_{71}(t) \le C + C\int_0^t \sup_{0\le r\le s}\mathbb{E}|x_r^n|^{p_0}ds, \tag{37}
$$

for any $t \in [0,T]$. In order to estimate $G_{72}(t)$, one writes

$$
\begin{aligned}
G_{72}(t) &= p_0(p_0-1)\mathbb{E}\int_0^t |x_{\kappa(n,s)}^n|^{p_0-2}\sum_{k=1}^d\sum_{v=1}^m \sigma^{n,(k,v)}(x_{\kappa(n,s)}^n)\sum_{j=1}^m\int_{\kappa(n,s)}^s L^{n,j}\sigma^{(k,v)}(x_{\kappa(n,r)}^n)\,dw_r^j\,ds \\
&+ p_0(p_0-1)\mathbb{E}\int_0^t |x_{\kappa(n,s)}^n|^{p_0-2}\sum_{k=1}^d\sum_{v=1}^m \sigma^{n,(k,v)}(x_{\kappa(n,s)}^n)\int_{\kappa(n,s)}^s L^{n,0}\sigma^{(k,v)}(x_{\kappa(n,r)}^n)\,dr\,ds \\
&+ p_0(p_0-1)\mathbb{E}\int_0^t |x_{\kappa(n,s)}^n|^{p_0-2}\sum_{k=1}^d\sum_{v=1}^m \sigma^{n,(k,v)}(x_{\kappa(n,s)}^n) \\
&\qquad\times\sum_{j=1}^m\sum_{j_1=1}^m\int_{\kappa(n,s)}^s\int_{\kappa(n,r)}^r L^{n,j}L^{j_1}\sigma^{(k,v)}(x_{\kappa(n,\gamma)}^n)\,dw_\gamma^{j_1}\,dw_r^j\,ds,
\end{aligned}
$$

which implies, due to Remark 3 and the fact that the first and third terms are zero,

$$
G_{72}(t) \le C\mathbb{E}\int_0^t \int_{\kappa(n,s)}^s n(1+|x_{\kappa(n,s)}^n|)^{p_0}\,dr\,ds,
$$

for any $t \in [0,T]$. Then, one obtains

$$
G_{72}(t) \le C + C\int_0^t \sup_{0\le r\le s}\mathbb{E}|x_r^n|^{p_0}ds, \tag{38}
$$

for any $t \in [0,T]$. Furthermore, substituting (37) and (38) into (36) yields

$$
G_7 \le C + C\int_0^t \sup_{0\le r\le s}\mathbb{E}|x_r^n|^{p_0}ds,
$$

for any $t \in [0,T]$. Therefore, for any $n \in \mathbb{N}$ and $t \in [0,T]$,

$$\sup_{0\le s\le t} \mathbb{E}|x_s^n|^{p_0} \le C + C\int_0^t \sup_{0\le r\le s} \mathbb{E}|x_r^n|^{p_0} ds < \infty,$$

and applying Gronwall's lemma completes the proof. $\square$

4 Proof of main result

Lemma 4. *Let $f : \mathbb{R}^d \to \mathbb{R}$ be a twice continuously differentiable function. If there exist constants $\alpha \in \mathbb{R}$, $K > 0$ and $\beta \in (0,1]$, such that for any $x, \bar{x} \in \mathbb{R}^d$,*

$$|\nabla^2 f(x) - \nabla^2 f(\bar{x})| \le K(1 + |x| + |\bar{x}|)^\alpha |x - \bar{x}|^\beta,$$

then, there is a constant $C > 0$ such that for any $x, \bar{x} \in \mathbb{R}^d$, and $i = 1, \ldots, d$,

$$\left| \frac{\partial f(x)}{\partial y^{(i)}} - \frac{\partial f(\bar{x})}{\partial y^{(i)}} - \sum_{j=1}^d \frac{\partial^2 f(\bar{x})}{\partial y^{(i)} \partial y^{(j)}} (x^{(j)} - \bar{x}^{(j)}) \right| \le C(1 + |x| + |\bar{x}|)^\alpha |x - \bar{x}|^{1+\beta}.$$

Proof. One uses the mean value theorem to obtain that, for all $x, \bar{x} \in \mathbb{R}^d$, $i = 1, \ldots, d$, there exists $q \in [0,1]$, such that

$$\frac{\partial f(x)}{\partial y^{(i)}} - \frac{\partial f(\bar{x})}{\partial y^{(i)}} = \sum_{j=1}^d \frac{\partial^2 f((qx + (1-q)\bar{x})}{\partial y^{(i)} \partial y^{(j)}} (x^{(j)} - \bar{x}^{(j)}).$$

Then for a fixed $q \in (0,1)$,

$$\begin{aligned}
&\left| \frac{\partial f(x)}{\partial y^{(i)}} - \frac{\partial f(\bar{x})}{\partial y^{(i)}} - \sum_{j=1}^d \frac{\partial^2 f(\bar{x})}{\partial y^{(i)} \partial y^{(j)}} (x^{(j)} - \bar{x}^{(j)}) \right| \\
&= \left| \sum_{j=1}^d \frac{\partial^2 f((qx + (1-q)\bar{x})}{\partial y^{(i)} \partial y^{(j)}} (x^{(j)} - \bar{x}^{(j)}) - \sum_{j=1}^d \frac{\partial^2 f(\bar{x})}{\partial y^{(i)} \partial y^{(j)}} (x^{(j)} - \bar{x}^{(j)}) \right| \\
&\le \sum_{j=1}^d \left| \frac{\partial^2 f((qx + (1-q)\bar{x})}{\partial y^{(i)} \partial y^{(j)}} - \frac{\partial^2 f(\bar{x})}{\partial y^{(i)} \partial y^{(j)}} \right| |x^{(j)} - \bar{x}^{(j)}| \\
&\le C(1 + |x| + |\bar{x}|)^\alpha |x - \bar{x}|^{1+\beta}.
\end{aligned}$$

$\square$

Lemma 5. *Assume **A-1** to **A-5** hold, then, there exists a constant $C > 0$, such that for any $p \le \frac{p_0}{2\rho+1}$ and $n \in \mathbb{N}$,*

$$\sup_{0\le t\le T} \mathbb{E}|b_1^n(t, x_{\kappa(n,t)}^n)|^p \le Cn^{-p}, \quad \sup_{0\le t\le T} \mathbb{E}|b_2^n(t, x_{\kappa(n,t)}^n)|^p \le Cn^{-\frac{p}{2}},$$

$$\sup_{0\le t\le T} \mathbb{E}|\sigma_1^n(t, x_{\kappa(n,t)}^n)|^p \le Cn^{-\frac{p}{2}}, \quad \sup_{0\le t\le T} \mathbb{E}|\sigma_2^n(t, x_{\kappa(n,t)}^n)|^p \le Cn^{-p},$$

$$\sup_{0\le t\le T} \mathbb{E}|\sigma_3^n(t, x_{\kappa(n,t)}^n)|^p \le Cn^{-p}.$$

Proof. By applying Hölder's inequality and Remark 1, one obtains, for any $p \leq \frac{p_0}{2\rho+1}$,

$$\begin{aligned}
\mathbb{E}|b_1^n(t, x^n_{\kappa(n,t)})|^p &= \mathbb{E}\left|\int_{\kappa(n,t)}^t L^{n,0}b(x^n_{\kappa(n,s)})\, ds\right|^p \\
&\leq Cn^{-p+1}\int_{\kappa(n,t)}^t \mathbb{E}|L^{n,0}b(x^n_{\kappa(n,s)})|^p\, ds \\
&\leq Cn^{-p+1}\int_{\kappa(n,t)}^t \mathbb{E}(1+|x^n_{\kappa(n,s)}|)^{(2\rho+1)p}\, ds \\
&\leq Cn^{-p},
\end{aligned}$$

where the last inequality holds due to Lemma 3. Other results can be proved by using similar arguments. □

Corollary 2. *Assume **A-1** to **A-5** hold, then, there exists a constant $C > 0$, such that for any $p \leq \frac{p_0}{2\rho+1}$ and $n \in \mathbb{N}$,*

$$\sup_{0\leq t\leq T}\mathbb{E}|\tilde{b}^n(t, x^n_{\kappa(n,t)})|^p \leq C, \quad \sup_{0\leq t\leq T}\mathbb{E}|\tilde{\sigma}^n(t, x^n_{\kappa(n,t)})|^p \leq C.$$

Lemma 6. *Assume **A-1** to **A-5** hold, then, there exists a constant $C > 0$, such that for any $p \leq \frac{p_0}{2\rho+1}$ and $n \in \mathbb{N}$,*

$$\sup_{0\leq t\leq T}\mathbb{E}|x^n_t - x^n_{\kappa(n,t)}|^p \leq Cn^{-\frac{p}{2}}.$$

Proof. For $p \geq 1$, by using Hölder's inequality, one obtains

$$\begin{aligned}
\mathbb{E}|x^n_t - x^n_{\kappa(n,t)}|^p &\leq C\mathbb{E}\left|\int_{\kappa(n,t)}^t \tilde{b}^n(s, x^n_{\kappa(n,s)})\, ds\right|^p + C\mathbb{E}\left|\int_{\kappa(n,t)}^t \tilde{\sigma}^n(s, x^n_{\kappa(n,s)})\, dw_s\right|^p \\
&\leq n^{-p+1}C\mathbb{E}\int_{\kappa(n,t)}^t |\tilde{b}^n(s, x^n_{\kappa(n,s)})|^p\, ds + Cn^{-\frac{p}{2}+1}\mathbb{E}\int_{\kappa(n,t)}^t |\tilde{\sigma}^n(s, x^n_{\kappa(n,s)})|^p\, ds,
\end{aligned}$$

which by using corollary 2 yields the desired result. As for $p \in (0, 1)$, one uses Jensen's inequality to obtain the same result. □

Lemma 7. *Assume **A-1** to **A-5** hold, then, there exists a constant $C > 0$, such that for any $p \leq \frac{p_0}{2\rho+1}$ and $n \in \mathbb{N}$,*

$$\sup_{0\leq t\leq T}\mathbb{E}|b(x^n_{\kappa(n,t)}) - b^n(x^n_{\kappa(n,t)})|^p \leq Cn^{-\frac{3}{2}p}, \quad \sup_{0\leq t\leq T}\mathbb{E}|\sigma(x^n_{\kappa(n,t)}) - \sigma^n(x^n_{\kappa(n,t)})|^p \leq Cn^{-\frac{3}{2}p}$$

Proof. We have the following expression,

$$|b(x^n_{\kappa(n,t)}) - b^n(x^n_{\kappa(n,t)})| = n^{-\frac{3}{2}}\frac{|x^n_{\kappa(n,t)}|^{3\rho}|b(x^n_{\kappa(n,t)})|}{1 + n^{-\frac{3}{2}}|x^n_{\kappa(n,t)}|^{3\rho}} \leq n^{-\frac{3}{2}}(1 + |x^n_{\kappa(n,t)}|)^{4\rho+1},$$

and then by using Lemma 3 and the same argument for σ completes the proof. □

Lemma 8. *Assume **A-1** to **A-5** hold and $p_0 \geq 2(5\rho+1)$. Then, there exists a constant $C > 0$, such that for any $n \in \mathbb{N}$,*

$$\sup_{0\leq t\leq T}\mathbb{E}|\sigma(x^n_t) - \sigma(x^n_{\kappa(n,t)}) - \sigma^n_M(t, x^n_{\kappa(n,t)})|^2 \leq Cn^{-(2+\beta)}.$$

Proof. For every $k=1,\ldots,d$, $v=1,\ldots,m$, applying Itô's formula to $\sigma^{(k,v)}(x_t^n)-\sigma^{(k,v)}(x_{\kappa(n,t)}^n)$ gives, almost surely,

$$
\begin{aligned}
&\sigma^{(k,v)}(x_t^n)-\sigma^{(k,v)}(x_{\kappa(n,t)}^n)\\
&=\sum_{i=1}^d\int_{\kappa(n,t)}^t\frac{\partial\sigma^{(k,v)}(x_s^n)}{\partial x^{(i)}}\tilde{b}^{n,(i)}(s,x_{\kappa(n,s)}^n)\,ds+\sum_{i=1}^d\sum_{j=1}^m\int_{\kappa(n,t)}^t\frac{\partial\sigma^{(k,v)}(x_s^n)}{\partial x^{(i)}}\tilde{\sigma}^{n,(i,j)}(s,x_{\kappa(n,s)}^n)\,dw_s^j\\
&\quad+\frac{1}{2}\sum_{i,l=1}^d\sum_{j=1}^m\int_{\kappa(n,t)}^t\frac{\partial^2\sigma^{(k,v)}(x_s^n)}{\partial x^{(i)}\partial x^{(l)}}\tilde{\sigma}^{n,(i,j)}(s,x_{\kappa(n,s)}^n)\tilde{\sigma}^{n,(l,j)}(s,x_{\kappa(n,s)}^n)\,ds=\sum_{i=1}^{12}J_i(t)
\end{aligned}
$$

where

$$
\begin{aligned}
J_1(t)&=\sum_{i=1}^d\int_{\kappa(n,t)}^t\left(\frac{\partial\sigma^{(k,v)}(x_s^n)}{\partial x^{(i)}}-\frac{\partial\sigma^{(k,v)}(x_{\kappa(n,s)}^n)}{\partial x^{(i)}}\right)b^{n,(i)}(x_{\kappa(n,s)}^n)\,ds,\\
J_2(t)&=\sum_{i=1}^d\int_{\kappa(n,t)}^t\frac{\partial\sigma^{(k,v)}(x_{\kappa(n,s)}^n)}{\partial x^{(i)}}b^{n,(i)}(x_{\kappa(n,s)}^n)\,ds,\\
J_3(t)&=\sum_{i=1}^d\int_{\kappa(n,t)}^t\frac{\partial\sigma^{(k,v)}(x_s^n)}{\partial x^{(i)}}(b_1^{n,(i)}(s,x_{\kappa(n,s)}^n)+b_2^{n,(i)}(s,x_{\kappa(n,s)}^n))\,ds,\\
J_4(t)&=\sum_{i=1}^d\sum_{j=1}^m\int_{\kappa(n,t)}^t\Bigg(\frac{\partial\sigma^{(k,v)}(x_s^n)}{\partial x^{(i)}}-\frac{\partial\sigma^{(k,v)}(x_{\kappa(n,s)}^n)}{\partial x^{(i)}}\\
&\qquad\qquad-\sum_{l=1}^d\frac{\partial^2\sigma^{(k,v)}(x_{\kappa(n,s)}^n)}{\partial x^{(i)}\partial x^{(l)}}(x_s^{n,(l)}-x_{\kappa(n,s)}^{n,(l)})\Bigg)\sigma^{n,(i,j)}(x_{\kappa(n,s)}^n)\,dw_s^j,\\
J_5(t)&=\sum_{i=1}^d\sum_{j=1}^m\int_{\kappa(n,t)}^t\sum_{l=1}^d\frac{\partial^2\sigma^{(k,v)}(x_{\kappa(n,s)}^n)}{\partial x^{(i)}\partial x^{(l)}}\Bigg(\int_{\kappa(n,s)}^s\tilde{b}^{n,(l)}(r,x_{\kappa(n,r)}^n)\,dr\\
&\qquad\qquad+\sum_{j_1=1}^m\int_{\kappa(n,s)}^s\sigma_M^{n,(l,j_1)}(r,x_{\kappa(n,r)}^n)\,dw_r^{j_1}\Bigg)\sigma^{n,(i,j)}(x_{\kappa(n,s)}^n)\,dw_s^j,\\
J_6(t)&=\sum_{i=1}^d\sum_{j=1}^m\int_{\kappa(n,t)}^t\sum_{l=1}^d\frac{\partial^2\sigma^{(k,v)}(x_{\kappa(n,s)}^n)}{\partial x^{(i)}\partial x^{(l)}}\sum_{j_1=1}^m\int_{\kappa(n,s)}^s\sigma^{n,(l,j_1)}(x_{\kappa(n,r)}^n)\,dw_r^{j_1}\sigma^{n,(i,j)}(x_{\kappa(n,s)}^n)\,dw_s^j,\\
J_7(t)&=\sum_{i=1}^d\sum_{j=1}^m\int_{\kappa(n,t)}^t\left(\frac{\partial\sigma^{(k,v)}(x_s^n)}{\partial x^{(i)}}-\frac{\partial\sigma^{(k,v)}(x_{\kappa(n,s)}^n)}{\partial x^{(i)}}\right)\sigma_1^{n,(i,j)}(s,x_{\kappa(n,s)}^n)\,dw_s^j,\\
J_8(t)&=\sum_{i=1}^d\sum_{j=1}^m\int_{\kappa(n,t)}^t\frac{\partial\sigma^{(k,v)}(x_{\kappa(n,s)}^n)}{\partial x^{(i)}}(\sigma^{n,(i,j)}(x_{\kappa(n,s)}^n)+\sigma_1^{n,(i,j)}(s,x_{\kappa(n,s)}^n))\,dw_s^j,\\
J_9(t)&=\sum_{i=1}^d\sum_{j=1}^m\int_{\kappa(n,t)}^t\frac{\partial\sigma^{(k,v)}(x_s^n)}{\partial x^{(i)}}(\sigma_2^{n,(i,j)}(s,x_{\kappa(n,s)}^n)+\sigma_3^{n,(i,j)}(s,x_{\kappa(n,s)}^n))\,dw_s^j,\\
J_{10}(t)&=\frac{1}{2}\sum_{i,l=1}^d\sum_{j=1}^m\int_{\kappa(n,t)}^t\left(\frac{\partial^2\sigma^{(k,v)}(x_s^n)}{\partial x^{(i)}\partial x^{(l)}}-\frac{\partial^2\sigma^{(k,v)}(x_{\kappa(n,s)}^n)}{\partial x^{(i)}\partial x^{(l)}}\right)\sigma^{n,(i,j)}(x_{\kappa(n,s)}^n)\sigma^{n,(l,j)}(x_{\kappa(n,s)}^n)\,ds,\\
J_{11}(t)&=\frac{1}{2}\sum_{i,l=1}^d\sum_{j=1}^m\int_{\kappa(n,t)}^t\frac{\partial^2\sigma^{(k,v)}(x_{\kappa(n,s)}^n)}{\partial x^{(i)}\partial x^{(l)}}\sigma^{n,(i,j)}(x_{\kappa(n,s)}^n)\sigma^{n,(l,j)}(x_{\kappa(n,s)}^n)\,ds,
\end{aligned}
$$

$$J_{12}(t) = \frac{1}{2}\sum_{i,l=1}^{d}\sum_{j=1}^{m}\int_{\kappa(n,t)}^{t} \frac{\partial^2\sigma^{(k,v)}(x_s^n)}{\partial x^{(i)}\partial x^{(l)}}(\sigma^{n,(i,j)}(x_{\kappa(n,s)}^n)\sigma_M^{n,(l,j)}(s, x_{\kappa(n,s)}^n) + \sigma_M^{n,(i,j)}(s, x_{\kappa(n,s)}^n)\tilde{\sigma}^{n,(l,j)}(s, x_{\kappa(n,s)}^n))\, ds.$$

It can be observed that

$$\begin{aligned}
&\mathbb{E}|J_2(t) + J_6(t) + J_8(t) + J_{11}(t) - \sigma_M^{n,(k,v)}(t, x_{\kappa(n,t)}^n)|^2 \\
&\leq 2\mathbb{E}|J_2(t) + J_{11}(t) - \sigma_2^{n,(k,v)}(t, x_{\kappa(n,t)}^n)|^2 + 2\mathbb{E}|J_6(t) + J_8(t) - \sigma_1^{n,(k,v)}(t, x_{\kappa(n,t)}^n) - \sigma_3^{n,(k,v)}(t, x_{\kappa(n,t)}^n)|^2 \\
&\leq C\sum_{i,l=1}^{d}\sum_{j=1}^{m}\mathbb{E}\left|-\frac{n^{-3/2}|x_{\kappa(n,t)}^n|^{3\rho}}{(1+n^{-3/2}|x_{\kappa(n,t)}^n|^{3\rho})^2}\int_{\kappa(n,t)}^{t}\frac{\partial^2\sigma^{(k,v)}(x_{\kappa(n,s)}^n)}{\partial x^{(i)}\partial x^{(l)}}\sigma^{(i,j)}(x_{\kappa(n,s)}^n)\sigma^{(l,j)}(x_{\kappa(n,s)}^n)\, ds\right|^2 \\
&\quad + 2\mathbb{E}\left|-\frac{n^{-3/2}|x_{\kappa(n,t)}^n|^{3\rho}}{(1+n^{-3/2}|x_{\kappa(n,t)}^n|^{3\rho})^2}\sum_{i,l=1}^{d}\sum_{j,j_1=1}^{m}\int_{\kappa(n,t)}^{t}\frac{\partial^2\sigma^{(k,v)}(x_{\kappa(n,s)}^n)}{\partial x^{(i)}\partial x^{(l)}}\right. \\
&\qquad\left.\times\int_{\kappa(n,s)}^{s}\sigma^{(l,j_1)}(x_{\kappa(n,r)}^n)\, dw_r^{j_1}\sigma^{(i,j)}(x_{\kappa(n,s)}^n)\, dw_s^{j}\right|^2,
\end{aligned}$$

which implies due to Remark 1 and Lemma 3 that

$$\begin{aligned}
&\mathbb{E}|J_2(t) + J_6(t) + J_8(t) + J_{11}(t) - \sigma_M^{n,(k,v)}(t, x_{\kappa(n,t)}^n)|^2 \\
&\leq Cn^{-3}\mathbb{E}|n^{-1}|x_{\kappa(n,t)}^n|^{3\rho}(1+|x_{\kappa(n,t)}^n|^{3/2\rho+1})|^2 + Cn^{-5}\mathbb{E}||x_{\kappa(n,t)}^n|^{3\rho}(1+|x_{\kappa(n,t)}^n|^{3/2\rho+1})|^2 \leq Cn^{-5},
\end{aligned}$$

for $p_0 \geq 9\rho + 2$. Then, one obtains the following

$$\begin{aligned}
&\mathbb{E}|\sigma^{(k,v)}(x_t^n) - \sigma^{(k,v)}(x_{\kappa(n,t)}^n) - \sigma_M^{n,(k,v)}(t, x_{\kappa(n,t)}^n)|^2 \\
&\quad\leq 2\mathbb{E}|J_1(t) + J_3(t) + J_4(t) + J_5(t) + J_7(t) + J_9(t) + J_{10}(t) + J_{12}(t)|^2 \\
&\qquad + 2\mathbb{E}|J_2(t) + J_6(t) + J_8(t) + J_{11}(t) - \sigma_M^{n,(k,v)}(t, x_{\kappa(n,t)}^n)|^2 \\
&\quad\leq C(\mathbb{E}|J_1(t)|^2 + \mathbb{E}|J_3(t)|^2 + \mathbb{E}|J_4(t)|^2 + \mathbb{E}|J_5(t)|^2 + \mathbb{E}|J_7(t)|^2 \\
&\qquad + \mathbb{E}|J_9(t)|^2 + \mathbb{E}|J_{10}(t)|^2 + \mathbb{E}|J_{12}(t)|^2) + Cn^{-5},
\end{aligned}$$

for any $t \in [0, T]$. By using Cauchy-Schwarz inequality, $\mathbb{E}|J_1(t)|^2$ can be estimated as

$$\mathbb{E}|J_1(t)|^2 \leq Cn^{-1}\sum_{i=1}^{d}\int_{\kappa(n,t)}^{t}\mathbb{E}\left|\left(\frac{\partial\sigma^{(k,v)}(x_s^n)}{\partial x^{(i)}} - \frac{\partial\sigma^{(k,v)}(x_{\kappa(n,s)}^n)}{\partial x^{(i)}}\right)b^n(x_{\kappa(n,s)}^n)\right|^2 ds,$$

which by using Young's inequality, Remark 1 and Hölder's inequality yields

$$\begin{aligned}
\mathbb{E}|J_1(t)|^2 &\leq Cn^{-1}\int_{\kappa(n,t)}^{t}\mathbb{E}(1+|x_s^n|+|x_{\kappa(n,s)}^n|)^{\rho-2}(1+|x_{\kappa(n,s)}^n|)^{2\rho+2}|x_s^n - x_{\kappa(n,s)}^n|^2\, ds \\
&\leq Cn^{-1}\int_{\kappa(n,t)}^{t}\mathbb{E}(1+|x_s^n|^{3\rho}+|x_{\kappa(n,s)}^n|^{3\rho})|x_s^n - x_{\kappa(n,s)}^n|^2\, ds \\
&\leq Cn^{-1}\int_{\kappa(n,t)}^{t}\left(\mathbb{E}(1+|x_s^n|^{p_0}+|x_{\kappa(n,s)}^n|^{p_0})\right)^{\frac{3\rho}{p_0}}\left(\mathbb{E}|x_s^n - x_{\kappa(n,s)}^n|^{\frac{2p_0}{p_0-3\rho}}\right)^{\frac{p_0-3\rho}{p_0}} ds,
\end{aligned}$$

for any $t \in [0, T]$. One uses Lemma 3 and Lemma 6 to obtain

$$\mathbb{E}|J_1(t)|^2 \leq Cn^{-3},$$

for every $n \in \mathbb{N}$. To estimate $\mathbb{E}|J_3(t)|^2$, one applies Cauchy-Schwarz inequality and Remark 1 to obtain

$$\mathbb{E}|J_3(t)|^2 \leq Cn^{-1} \int_{\kappa(n,t)}^{t} \mathbb{E}(1+|x_s^n|)^{\rho}(|b_1^n(s,x_{\kappa(n,s)}^n)|^2 + |b_2^n(s,x_{\kappa(n,s)}^n)|^2)\, ds,$$

which implies due to Hölder's inequality

$$\mathbb{E}|J_3(t)|^2 \leq Cn^{-1} \int_{\kappa(n,t)}^{t} (\mathbb{E}(1+|x_s^n|^{p_0}))^{\frac{\rho}{p_0}} (\mathbb{E}(|b_1^n(s,x_{\kappa(n,s)}^n)|^{\frac{2p_0}{p_0-\rho}} + |b_2^n(s,x_{\kappa(n,s)}^n)|^{\frac{2p_0}{p_0-\rho}}))^{\frac{p_0-\rho}{p_0}}\, ds,$$

for any $t \in [0,T]$. By Lemma 3 and Lemma 5, it becomes

$$\mathbb{E}|J_3(t)|^2 \leq Cn^{-3},$$

for every $n \in \mathbb{N}$. As for $\mathbb{E}|J_4(t)|^2$, by using Young's inequality, Cauchy-Schwarz inequality, Remark 1 and Lemma 4, one obtains

$$\begin{aligned}
\mathbb{E}|J_4(t)|^2 &\leq C \int_{\kappa(n,t)}^{t} \mathbb{E}((1+|x_s^n|+|x_{\kappa(n,s)}^n|)^{\rho-4}(1+|x_{\kappa(n,s)}^n|)^{\rho+2}|x_s^n - x_{\kappa(n,s)}^n|^{2+2\beta}\, ds \\
&\leq C \int_{\kappa(n,t)}^{t} \mathbb{E}((1+|x_s^n|+|x_{\kappa(n,s)}^n|)^{2\rho-2}|x_s^n - x_{\kappa(n,s)}^n|^{2+2\beta})\, ds,
\end{aligned}$$

which implies due to Hölder's inequality

$$\mathbb{E}|J_4(t)|^2 \leq C \int_{\kappa(n,t)}^{t} \Big(\mathbb{E}(1+|x_s^n|^{p_0}+|x_{\kappa(n,s)}^n|^{p_0})\Big)^{\frac{2\rho-2}{p_0}} \Big(\mathbb{E}|x_s^n - x_{\kappa(n,s)}^n|^{\frac{(2+2\beta)p_0}{p_0-2\rho+2}}\Big)^{\frac{p_0-2\rho+2}{p_0}}\, ds,$$

for any $t \in [0,T]$. Then, applying Lemma 6 and Lemma 3 yield

$$\mathbb{E}|J_4(t)|^2 \leq Cn^{-(2+\beta)},$$

for every $n \in \mathbb{N}$. In order to estimate $\mathbb{E}|J_5(t)|^2$, one uses Young's inequality and Cauchy-Schwarz inequality to obtain

$$\begin{aligned}
\mathbb{E}|J_5(t)|^2 \leq C \int_{\kappa(n,t)}^{t} \mathbb{E}\Bigg|\int_{\kappa(n,s)}^{s} |\tilde{b}^n(r,x_{\kappa(n,r)}^n)|\, dr + \Bigg|\sum_{j_1=1}^{m} \int_{\kappa(n,s)}^{s} \sigma_M^{n,(i,j_1)}(r,x_{\kappa(n,r)}^n)\, dw_r^{j_1}\Bigg|\Bigg|^2 \\
\times (1+|x_{\kappa(n,s)}^n|)^{2\rho}\, ds,
\end{aligned}$$

which, by applying Hölder's inequality, yields

$$\begin{aligned}
\mathbb{E}|J_5(t)|^2 \leq C \int_{\kappa(n,t)}^{t} \Bigg(n^{-\frac{2p_0}{p_0-2\rho}+1} \int_{\kappa(n,s)}^{s} \mathbb{E}|\tilde{b}^n(r,x_{\kappa(n,r)}^n)|^{\frac{2p_0}{p_0-2\rho}}\, ds \\
+ n^{-\frac{p_0}{p_0-2\rho}+1} \int_{\kappa(n,s)}^{s} \mathbb{E}|\sigma_M^n(r,x_{\kappa(n,r)}^n)|^{\frac{2p_0}{p_0-2\rho}}\, ds\Bigg)^{\frac{p_0-2\rho}{p_0}} \Big(\mathbb{E}(1+|x_{\kappa(n,s)}^n|^{p_0})\Big)^{\frac{2\rho}{p_0}}\, ds,
\end{aligned}$$

for any $t \in [0,T]$. One uses Corollary 2 and Lemma 5 to obtain

$$\mathbb{E}|J_5(t)|^2 \leq Cn^{-3},$$

for every $n \in \mathbb{N}$. As for $\mathbb{E}|J_7(t)|^2$, it can be estimated by using Cauchy-Schwarz inequality as follows

$$\mathbb{E}|J_7(t)|^2 \leq C \sum_{i=1}^{d} \sum_{j=1}^{m} \int_{\kappa(n,t)}^{t} \mathbb{E}\left|\frac{\partial \sigma^{(k,v)}(x_s^n)}{\partial x^{(i)}} - \frac{\partial \sigma^{(k,v)}(x_{\kappa(n,s)}^n)}{\partial x^{(i)}}\right|^2 |\sigma_1^n(s, x_{\kappa(n,s)}^n)|^2 \, ds,$$

which yields by using Remark 1 and Hölder's inequality

$$\begin{aligned}
\mathbb{E}|J_7(t)|^2 &\leq C \int_{\kappa(n,t)}^{t} \mathbb{E}(1 + |x_s^n| + |x_{\kappa(n,s)}^n|)^{\rho-2} |x_s^n - x_{\kappa(n,s)}^n|^2 |\sigma_1^n(s, x_{\kappa(n,s)}^n)|^2 \, ds \\
&\leq C \int_{\kappa(n,t)}^{t} \left(\mathbb{E}(1 + |x_s^n| + |x_{\kappa(n,s)}^n|)^{p_0}\right)^{\frac{\rho-2}{p_0}} \\
&\qquad \times \left(\mathbb{E}|x_s^n - x_{\kappa(n,s)}^n)|^{\frac{2p_0}{p_0-\rho+2}} |\sigma_1^{n,}(s, x_{\kappa(n,s)}^n)|^{\frac{2p_0}{p_0-\rho+2}}\right)^{\frac{p_0-\rho+2}{p_0}} \, ds,
\end{aligned}$$

for $\rho > 2$, and any $t \in [0, T]$. Then, one can apply Cauchy-Schwarz inequality and Lemma 3 to obtain

$$\mathbb{E}|J_7(t)|^2 \leq C \int_{\kappa(n,t)}^{t} \left(\mathbb{E}|x_s^n - x_{\kappa(n,s)}^n)|^{\frac{4p_0}{p_0-\rho+2}} \mathbb{E}|\sigma_1^{n,}(s, x_{\kappa(n,s)}^n)|^{\frac{4p_0}{p_0-\rho+2}}\right)^{\frac{p_0-\rho+2}{2p_0}} \, ds,$$

Thus, applying Lemma 6 and Lemma 5 give the following estimate

$$\mathbb{E}|J_7(t)|^2 \leq C n^{-3},$$

for every $n \in \mathbb{N}$. Note that, for the case that $\rho = 2$, one obtains the same result immediately by using Cauchy-Schwarz inequality. As for $\mathbb{E}|J_9(t)|^2$, applying Remark 1 yields

$$\mathbb{E}|J_9(t)|^2 \leq C \int_{\kappa(n,t)}^{t} \mathbb{E}(1 + |x_s^n|)^{\rho} (|\sigma_2^n(s, x_{\kappa(n,s)}^n)|^2 + |\sigma_3^n(s, x_{\kappa(n,s)}^n)|^2) \, ds$$

which by applying Hölder's inequality gives

$$\mathbb{E}|J_9(t)|^2 \leq C \int_{\kappa(n,t)}^{t} (\mathbb{E}(1 + |x_s^n|^{p_0}))^{\frac{\rho}{p_0}} \left(\mathbb{E}(|\sigma_2^n(s, x_{\kappa(n,s)}^n)|^{\frac{2p_0}{p_0-\rho}} + |\sigma_3^n(s, x_{\kappa(n,s)}^n)|^{\frac{2p_0}{p_0-\rho}})\right)^{\frac{p_0-\rho}{p_0}} \, ds,$$

for any $t \in [0, T]$. By Lemma 5, one obtains

$$\mathbb{E}|J_9(t)|^2 \leq C n^{-3},$$

for every $n \in \mathbb{N}$. To estimate $\mathbb{E}|J_{10}(t)|^2$, one uses Young's inequality and Remark 1 to obtain

$$\mathbb{E}|J_{10}(t)|^2 \leq C n^{-1} \int_{\kappa(n,t)}^{t} \mathbb{E}(1 + |x_s^n| + |x_{\kappa(n,s)}^n|)^{3\rho} |x_s^n - x_{\kappa(n,s)}^n|^{2\beta} \, ds$$

which implies due to Hölder's inequality

$$\mathbb{E}|J_{10}(t)|^2 \leq C n^{-1} \int_{\kappa(n,t)}^{t} \left(\mathbb{E}(1 + |x_{\kappa(n,s)}^n|)^{p_0}\right)^{\frac{3\rho}{p_0}} \left(\mathbb{E}|x_s^n - x_{\kappa(n,s)}^n|^{\frac{2\beta p_0}{p_0-3\rho}}\right)^{\frac{p_0-3\rho}{p_0}} \, ds,$$

for any $t \in [0, T]$. Lemma 6 is used to obtain

$$\mathbb{E}|J_{10}(t)|^2 \leq C n^{-(2+\beta)},$$

for every $n\in\mathbb{N}$. Finally for $\mathbb{E}|J_{12}(t)|^2$, applying Young's inequality, Cauchy-Schwarz inequality and Remark 1 yield

$$\mathbb{E}|J_{12}(t)|^2 \leq Cn^{-1}\int_{\kappa(n,t)}^{t} \mathbb{E}(1+|x_s^n|)+|x_{\kappa(n,s)}^n|)^{2\rho}|\sigma_M^n(s,x_{\kappa(n,s)}^n)|^2\,ds$$
$$+Cn^{-1}\int_{\kappa(n,t)}^{t} \mathbb{E}(1+|x_s^n|)^{\rho-2}|\tilde{\sigma}^n(x_{\kappa(n,s)}^n)|^2|\sigma_M^n(s,x_{\kappa(n,s)}^n)|^2\,ds,$$

which implies due to Hölder's inequality

$$\mathbb{E}|J_{12}(t)|^2 \leq Cn^{-1}\int_{\kappa(n,t)}^{t}\Big(\mathbb{E}(1+|x_s^n|^{p_0}+|x_{\kappa(n,s)}^n|^{p_0})\Big)^{\frac{2\rho}{p_0}}\left(\mathbb{E}|\sigma_M^n(s,x_{\kappa(n,s)}^n)|^{\frac{2p_0}{p_0-2\rho}}\right)^{\frac{p_0-2\rho}{p_0}}ds$$
$$+Cn^{-1}\int_{\kappa(n,t)}^{t}(\mathbb{E}(1+|x_s^n|)^{p_0})^{\frac{\rho-2}{p_0}}\left(\mathbb{E}|\tilde{\sigma}^n(x_{\kappa(n,s)}^n)|^{\frac{2p_0}{p_0-\rho+2}}|\sigma_M^n(x_{\kappa(n,s)}^n)|^{\frac{2p_0}{p_0-\rho+2}}\right)^{\frac{p_0-\rho+2}{p_0}}ds,$$

for any $t\in[0,T]$. Applying Lemma 3 and Lemma 5 to the first term and Cauchy-Schwarz inequality to the second term give

$$\mathbb{E}|J_{12}(t)|^2 \leq Cn^{-3}+Cn^{-1}\int_{\kappa(n,t)}^{t}\left(\mathbb{E}|\tilde{\sigma}^n(x_{\kappa(n,s)}^n)|^{\frac{4p_0}{p_0-\rho+2}}\mathbb{E}|\sigma_M^n(x_{\kappa(n,s)}^n)|^{\frac{4p_0}{p_0-\rho+2}}\right)^{\frac{p_0-\rho+2}{2p_0}}ds,$$

which by using Lemma 5 yields the desired result, i.e.

$$\mathbb{E}|J_{12}(t)|^2\leq Cn^{-3},$$

for every $n\in\mathbb{N}$. Therefore, one obtains, for any $n\in\mathbb{N}$, $\beta\in(0,1]$ and $p_0\geq 10\rho+2$,

$$\sup_{0\leq t\leq T}\mathbb{E}|\sigma(x_t^n)-\sigma(x_{\kappa(n,t)}^n)-\sigma_M^n(t,x_{\kappa(n,t)}^n)|^2\leq Cn^{-(2+\beta)}+Cn^{-3}+Cn^{-5}\leq Cn^{-(2+\beta)}.$$

$\square$

Lemma 9. *Assume **A-1** to **A-5** hold and $p_0\geq 2(5\rho+1)$. Then, there exists a constant $C>0$, such that for any $n\in\mathbb{N}$,*

$$\sup_{0\leq t\leq T}\mathbb{E}|b(x_t^n)-b(x_{\kappa(n,t)}^n)-b_1^n(t,x_{\kappa(n,t)}^n)-b_2^n(t,x_{\kappa(n,t)}^n)|^2\leq Cn^{-2}.$$

Proof. For every $k=1,\ldots,d$, applying Itô's formula to $b^{(k)}(x_t^n)-b^{(k)}(x_{\kappa(n,t)}^n)$ gives, almost surely,

$$\begin{aligned}
&b^{(k)}(x_t^n)-b^{(k)}(x_{\kappa(n,t)}^n)\\
&=\sum_{i=1}^{d}\int_{\kappa(n,t)}^{t}\frac{\partial b^{(k)}(x_s^n)}{\partial x^{(i)}}\tilde{b}^{n,(i)}(s,x_{\kappa(n,s)}^n)\,ds+\sum_{i=1}^{d}\sum_{j=1}^{m}\int_{\kappa(n,t)}^{t}\frac{\partial b^{(k)}(x_s^n)}{\partial x^{(i)}}\tilde{\sigma}^{n,(i,j)}(s,x_{\kappa(n,s)}^n)\,dw_s^j\\
&\quad+\frac{1}{2}\sum_{i,l=1}^{d}\sum_{j=1}^{m}\int_{\kappa(n,t)}^{t}\frac{\partial^2 b^{(k)}(x_s^n)}{\partial x^{(i)}\partial x^{(l)}}\tilde{\sigma}^{n,(i,j)}(s,x_{\kappa(n,s)}^n)\tilde{\sigma}^{n,(l,j)}(s,x_{\kappa(n,s)}^n)\,ds\\
&=\sum_{i=1}^{9}I_i(t),
\end{aligned}\tag{41}$$

where

$$I_1(t)=\sum_{i=1}^{d}\int_{\kappa(n,t)}^{t}\left(\frac{\partial b^{(k)}(x_s^n)}{\partial x^{(i)}}-\frac{\partial b^{(k)}(x_{\kappa(n,s)}^n)}{\partial x^{(i)}}\right)b^{n,(i)}(x_{\kappa(n,s)}^n)\,ds,$$

$$I_2(t)=\sum_{i=1}^{d}\int_{\kappa(n,t)}^{t}\frac{\partial b^{(k)}(x_{\kappa(n,s)}^n)}{\partial x^{(i)}}b^{n,(i)}(x_{\kappa(n,s)}^n)\,ds,$$

$$I_3(t)=\sum_{i=1}^{d}\int_{\kappa(n,t)}^{t}\frac{\partial b^{(k)}(x_s^n)}{\partial x^{(i)}}(b_1^{n,(i)}(s,x_{\kappa(n,s)}^n)+b_2^{n,(i)}(s,x_{\kappa(n,s)}^n))\,ds,$$

$$I_4(t)=\sum_{i=1}^{d}\sum_{j=1}^{m}\int_{\kappa(n,t)}^{t}\left(\frac{\partial b^{(k)}(x_s^n)}{\partial x^{(i)}}-\frac{\partial b^{(k)}(x_{\kappa(n,s)}^n)}{\partial x^{(i)}}\right)\sigma^{n,(i,j)}(x_{\kappa(n,s)}^n)\,dw_s^j,$$

$$I_5(t)=\sum_{i=1}^{d}\sum_{j=1}^{m}\int_{\kappa(n,t)}^{t}\frac{\partial b^{(k)}(x_{\kappa(n,s)}^n)}{\partial x^{(i)}}\sigma^{n,(i,j)}(x_{\kappa(n,s)}^n)\,dw_s^j,$$

$$I_6(t)=\sum_{i=1}^{d}\sum_{j=1}^{m}\int_{\kappa(n,t)}^{t}\frac{\partial b^{(k)}(x_s^n)}{\partial x^{(i)}}\sigma_M^{n,(i,j)}(s,x_{\kappa(n,s)}^n)\,dw_s^j,$$

$$I_7(t)=\frac{1}{2}\sum_{i,l=1}^{d}\sum_{j=1}^{m}\int_{\kappa(n,t)}^{t}\left(\frac{\partial^2 b^{(k)}(x_s^n)}{\partial x^{(i)}\partial x^{(l)}}-\frac{\partial^2 b^{(k)}(x_{\kappa(n,s)}^n)}{\partial x^{(i)}\partial x^{(l)}}\right)\sigma^{n,(i,j)}(x_{\kappa(n,s)}^n)\sigma^{n,(l,j)}(x_{\kappa(n,s)}^n)\,ds,$$

$$I_8(t)=\frac{1}{2}\sum_{i,l=1}^{d}\sum_{j=1}^{m}\int_{\kappa(n,t)}^{t}\frac{\partial^2 b^{(k)}(x_{\kappa(n,s)}^n)}{\partial x^{(i)}\partial x^{(l)}}\sigma^{n,(i,j)}(x_{\kappa(n,s)}^n)\sigma^{n,(l,j)}(x_{\kappa(n,s)}^n)\,ds,$$

$$\begin{aligned}I_9(t)=\frac{1}{2}\sum_{i,l=1}^{d}\sum_{j=1}^{m}\int_{\kappa(n,t)}^{t}\frac{\partial^2 b^{(k)}(x_s^n)}{\partial x^{(i)}\partial x^{(l)}}(&\sigma^{n,(i,j)}(x_{\kappa(n,s)}^n)\sigma_M^{n,(l,j)}(s,x_{\kappa(n,s)}^n)\\&+\sigma_M^{n,(i,j)}(s,x_{\kappa(n,s)}^n)\tilde{\sigma}^{n,(l,j)}(s,x_{\kappa(n,s)}^n))\,ds.\end{aligned}$$

Note that

$$\begin{aligned}&\mathbb{E}|I_2(t)+I_8(t)-b_1^{n,(k)}(t,x_{\kappa(n,t)}^n)|^2\\&\leq C\sum_{i,l=1}^{d}\sum_{j=1}^{m}\mathbb{E}\left|-\frac{n^{-3/2}|x_{\kappa(n,t)}^n|^{3\rho}}{(1+n^{-3/2}|x_{\kappa(n,t)}^n|^{3\rho})^2}\int_{\kappa(n,t)}^{t}\frac{\partial^2 b^{(k)}(x_{\kappa(n,s)}^n)}{\partial x^{(i)}\partial x^{(l)}}\sigma^{(i,j)}(x_{\kappa(n,s)}^n)\sigma^{(l,j)}(x_{\kappa(n,s)}^n)\,ds\right|^2,\end{aligned}$$

which by applying Remark 1 and Lemma 3 yields

$$\mathbb{E}|I_2(t)+I_8(t)-b_1^{n,(k)}(t,x_{\kappa(n,t)}^n)|^2\leq Cn^{-5}\mathbb{E}||x_{\kappa(n,t)}^n|^{3\rho}(1+|x_{\kappa(n,t)}^n|^{2\rho+1})|^2\leq Cn^{-5},\tag{42}$$

for $p_0\geq 10\rho+2$. Moreover, notice that

$$I_5(t)=b_2^{n,(k)}(t,x_{\kappa(n,t)}^n).\tag{43}$$

Then, one obtains the following

$$\begin{aligned}&\mathbb{E}|b^{(k)}(x_t^n)-b^{(k)}(x_{\kappa(n,t)}^n)-b_1^{n,(k)}(t,x_{\kappa(n,t)}^n)-b_2^{n,(k)}(t,x_{\kappa(n,t)}^n)|^2\\&\quad\leq 2\mathbb{E}|I_1(t)+I_3(t)+I_4(t)+I_6(t)+I_7(t)+I_9(t)|^2+2\mathbb{E}|I_2(t)+I_8(t)-b_1^{n,(k)}(t,x_{\kappa(n,t)}^n)|^2\\&\quad\leq C(\mathbb{E}|I_1(t)|^2+\mathbb{E}|I_3(t)|^2+\mathbb{E}|I_4(t)|^2+\mathbb{E}|I_6(t)|^2+\mathbb{E}|I_7(t)|^2+\mathbb{E}|I_9(t)|^2)+Cn^{-5},\end{aligned}$$

for any $t \in [0,T]$. To estimate $\mathbb{E}|I_1(t)|^2$, applying Cauchy-Schwarz inequality and Remark 1 yield

$$\mathbb{E}|I_1(t)|^2 \leq Cn^{-1}\int_{\kappa(n,t)}^{t} \mathbb{E}(1+|x_s^n|+|x_{\kappa(n,s)}^n|)^{2\rho-2}(1+|x_{\kappa(n,s)}^n|)^{2\rho+2}|x_s^n - x_{\kappa(n,s)}^n|^2\, ds,$$

which further implies due to Young's inequality and Hölder's inequality

$$\mathbb{E}|I_1(t)|^2 \leq Cn^{-1}\int_{\kappa(n,t)}^{t} \left(\mathbb{E}(1+|x_s^n|^{p_0}+|x_{\kappa(n,s)}^n|^{p_0})\right)^{\frac{4\rho}{p_0}} \left(\mathbb{E}|x_s^n - x_{\kappa(n,s)}^n|^{\frac{2p_0}{p_0-4\rho}}\right)^{\frac{p_0-4\rho}{p_0}} ds,$$

for any $t \in [0,T]$. By Lemma 6, one obtains

$$\mathbb{E}|I_1(t)|^2 \leq Cn^{-3},$$

for any $n \in \mathbb{N}$. As for $\mathbb{E}|I_3(t)|^2$, applying Cauchy-Schwarz inequality and Remark 1 give

$$\mathbb{E}|I_3(t)|^2 \leq Cn^{-1}\int_{\kappa(n,t)}^{t} \mathbb{E}(1+|x_s^n|)^{2\rho}(|b_1^n(s,x_{\kappa(n,s)}^n)|^2+|b_2^n(s,x_{\kappa(n,s)}^n)|^2)\, ds,$$

then one writes by using Hölder's inequality that

$$\mathbb{E}|I_3(t)|^2 \leq Cn^{-1}\int_{\kappa(n,t)}^{t} (\mathbb{E}(1+|x_s^n|^{p_0}))^{\frac{2\rho}{p_0}} (\mathbb{E}|b_1^n(s,x_{\kappa(n,s)}^n)|^{\frac{2p_0}{p_0-2\rho}}+\mathbb{E}|b_2^n(s,x_{\kappa(n,s)}^n)|^{\frac{2p_0}{p_0-2\rho}})^{\frac{p_0-2\rho}{p_0}} ds,$$

for any $t \in [0,T]$. Applying Lemma 5 yields

$$\mathbb{E}|I_3(t)|^2 \leq Cn^{-3},$$

for any $n \in \mathbb{N}$. To estimate $\mathbb{E}|I_4(t)|^2$, one uses Cauchy-Schwarz inequality, Remark 1 and Young's inequality to obtain

$$\begin{aligned}\mathbb{E}|I_4(t)|^2 &\leq C\int_{\kappa(n,t)}^{t} \mathbb{E}(1+|x_s^n|+|x_{\kappa(n,s)}^n|)^{2\rho-2}(1+|x_{\kappa(n,s)}^n|)^{\rho+2}|x_s^n - x_{\kappa(n,s)}^n|^2\, ds \\ &\leq C\int_{\kappa(n,t)}^{t} \mathbb{E}(1+|x_s^n|+|x_{\kappa(n,s)}^n|)^{3\rho}|x_s^n - x_{\kappa(n,s)}^n|^2\, ds,\end{aligned}$$

which implies due to Young's inequality and Hölder's inequality

$$\mathbb{E}|I_4(t)|^2 \leq C\int_{\kappa(n,t)}^{t} \left(\mathbb{E}(1+|x_s^n|^{p_0}+|x_{\kappa(n,s)}^n|^{p_0})\right)^{\frac{3\rho}{p_0}} \left(\mathbb{E}|x_s^n - x_{\kappa(n,s)}^n|^{\frac{2p_0}{p_0-3\rho}}\right)^{\frac{p_0-3\rho}{p_0}} ds,$$

for any $t \in [0,T]$. One applies Lemma 6 to obtain

$$\mathbb{E}|I_4(t)|^2 \leq Cn^{-2}, \tag{44}$$

for any $n \in \mathbb{N}$. As for $\mathbb{E}|I_6(t)|^2$, it can be written as

$$\mathbb{E}|I_6(t)|^2 \leq C\int_{\kappa(n,t)}^{t} \mathbb{E}(1+|x_s^n|)^{2\rho}|\sigma_M^n(s,x_{\kappa(n,s)}^n)|^2\, ds,$$

which by using Hölder's inequality yields

$$\mathbb{E}|I_6(t)|^2 \leq C\int_{\kappa(n,t)}^{t} (\mathbb{E}(1+|x_s^n|^{p_0}))^{\frac{2\rho}{p_0}} \left(\mathbb{E}|\sigma_M^n(s,x_{\kappa(n,s)}^n)|^{\frac{2p_0}{p_0-2\rho}}\right)^{\frac{p_0-2\rho}{p_0}} ds,$$

for any $t \in [0,T]$. By using Lemma 3 and Lemma 5, one obtains

$$\mathbb{E}|I_6(t)|^2 \leq Cn^{-2}, \tag{45}$$

for any $n \in \mathbb{N}$. In order to estimate $\mathbb{E}|I_7(t)|^2$, one uses Cauchy-Schwarz inequality and Remark 1 to obtain

$$\begin{aligned}
\mathbb{E}|I_7(t)|^2 &\leq Cn^{-1} \int_{\kappa(n,t)}^{t} \mathbb{E}(1+|x_s^n|+|x_{\kappa(n,s)}^n|)^{2\rho-4}(1+|x_{\kappa(n,s)}^n|)^{2\rho+4}|x_s^n - x_{\kappa(n,s)}^n|^2\, ds \\
&\leq Cn^{-1} \int_{\kappa(n,t)}^{t} \mathbb{E}(1+|x_s^n|+|x_{\kappa(n,s)}^n|)^{4\rho}|x_s^n - x_{\kappa(n,s)}^n|^2\, ds,
\end{aligned}$$

which by applying Young's inequality and Hölder's inequality yields

$$\mathbb{E}|I_7(t)|^2 \leq Cn^{-1} \int_{\kappa(n,t)}^{t} \left(\mathbb{E}(1+|x_s^n|+|x_{\kappa(n,s)}^n|)^{p_0}\right)^{\frac{4\rho}{p_0}} \left(\mathbb{E}|x_s^n - x_{\kappa(n,s)}^n|^{\frac{2p_0}{p_0-4\rho}}\right)^{\frac{p_0-4\rho}{p_0}} ds,$$

for any $t \in [0,T]$. Then applying Lemma 3 and Lemma 6, one obtains

$$\mathbb{E}|I_7(t)|^2 \leq Cn^{-3},$$

for any $n \in \mathbb{N}$. Finally for $\mathbb{E}|I_9(t)|^2$, one writes

$$\begin{aligned}
\mathbb{E}|I_9(t)|^2 &\leq Cn^{-1} \int_{\kappa(n,t)}^{t} \mathbb{E}(1+|x_s^n|)^{2\rho-2}(1+|x_{\kappa(n,s)}^n|)^{\rho+2}|\sigma_M^n(s,x_{\kappa(n,s)}^n)|^2\, ds \\
&\quad + Cn^{-1} \int_{\kappa(n,t)}^{t} \mathbb{E}(1+|x_s^n|)^{2\rho-2}|\sigma_M^n(s,x_{\kappa(n,s)}^n)|^2|\tilde{\sigma}^n(s,x_{\kappa(n,s)}^n)|^2\, ds,
\end{aligned}$$

which implies due to Young's inequality and Hölder's inequality

$$\begin{aligned}
\mathbb{E}|I_9(t)|^2 &\leq Cn^{-1} \int_{\kappa(n,t)}^{t} \left(\mathbb{E}(1+|x_s^n|+|x_{\kappa(n,s)}^n|)^{p_0}\right)^{\frac{3\rho}{p_0}} \left(\mathbb{E}|\sigma_M^n(s,x_{\kappa(n,s)}^n)|^{\frac{2p_0}{p_0-3\rho}}\right)^{\frac{p_0-3\rho}{p_0}} ds \\
&\quad + Cn^{-1} \int_{\kappa(n,t)}^{t} (\mathbb{E}(1+|x_s^n|)^{p_0})^{\frac{2\rho-2}{p_0}} \left(\mathbb{E}|\tilde{\sigma}^n(s,x_{\kappa(n,s)}^n)|^{\frac{2p_0}{p_0-2\rho+2}}|\sigma_M^n(s,x_{\kappa(n,s)}^n)|^{\frac{2p_0}{p_0-2\rho+2}}\right)^{\frac{p_0-2\rho+2}{p_0}} ds
\end{aligned}$$

for any $t \in [0,T]$. One can then apply Lemma 3 and Lemma 5 to the first term, and apply Cauchy-Schwarz inequality to the second term to obtain

$$\mathbb{E}|I_9(t)|^2 \leq Cn^{-3} + Cn^{-1} \int_{\kappa(n,t)}^{t} \left(\mathbb{E}|\tilde{\sigma}^n(s,x_{\kappa(n,s)}^n)|^{\frac{4p_0}{p_0-2\rho+2}}\mathbb{E}|\sigma_M^n(s,x_{\kappa(n,s)}^n)|^{\frac{4p_0}{p_0-2\rho+2}}\right)^{\frac{p_0-2\rho+2}{2p_0}} ds$$

which, by using Lemma 5, implies

$$\mathbb{E}|I_9(t)|^2 \leq Cn^{-3},$$

for any $n \in \mathbb{N}$ and $t \in [0,T]$. Therefore,

$$\sup_{0\leq t\leq T} \mathbb{E}|b(x_t^n) - b(x_{\kappa(n,t)}^n) - b_1^n(t,x_{\kappa(n,t)}^n) - b_2^n(t,x_{\kappa(n,t)}^n)|^2 \leq Cn^{-2} + Cn^{-5} \leq Cn^{-2},$$

for any $n \in \mathbb{N}$, and the proof is complete. □

Denote by $e_t^n := x_t - x_t^n$ for any $t \in [0,T]$, and define the stopping times as follows: for $R > 0$,

$$.\tau_R := \inf\{t \geq 0 : |x_t| \geq R\}, \quad \tau'_{n,R} := \inf\{t \geq 0 : |x_t^n| \geq R\}, \quad \nu_{n,R} := \tau_R \wedge \tau'_{n,R}. \tag{46}$$

Lemma 10. *Assume* ***A-1*** *to* ***A-5*** *hold and $p_0 \geq 2(5\rho+1)$. Then, there exists a constant $C > 0$ such that for any $s \in [0,T]$, the following inequality holds*

$$\mathbb{P}(s > \nu_{n,R}) \leq CR^{-2},$$

where $\nu_{n,R}$ is the stopping time defined in (46).

Proof. By applying Markov inequality, one obtains

$$\begin{aligned}\mathbb{P}(s > \nu_{n,R}) &\leq \mathbb{P}\left(\sup_{u\leq s}|x_u| > R\right) + \mathbb{P}\left(\sup_{u\leq s}|x_u^n| > R\right) \\ &\leq R^{-2}\mathbb{E}\left(\sup_{u\leq s}|x_u|^2\right) + R^{-2}\mathbb{E}\left(\sup_{u\leq s}|x_u^n|^2\right) \\ &\leq CR^{-2}.\end{aligned}$$

Note that the last inequality holds since by Lemma 1 and Lemma 3, we have shown that the p_0-th moment of x_t and x_t^n are bounded uniformly in time, i.e. $\sup_{0\leq t\leq T}\mathbb{E}|x_t|^{p_0} \leq C$ and $\sup_{0\leq t\leq T}\mathbb{E}|x_t^n|^{p_0} \leq C$ for all $n \in \mathbb{N}$ and $p_0 \geq 4$. Then, one can obtain the uniform $\mathcal{L}^2$ bound by using Lemma 5 in [15], which originally appeared in [5]. □

Lemma 11. *Assume* ***A-1*** *to* ***A-5*** *hold and $p_0 \geq 2(5\rho+1)$. Then, there exists a constant $C > 0$, which is independent of R, such that for any $n \in \mathbb{N}$ and $t \in [0,T]$,*

$$\begin{aligned}\mathbb{E}\int_0^{t\wedge\nu_{n,R}} e_s^n(b(x_s^n) - b(x_{\kappa(n,s)}^n) - b_1^n(s,x_{\kappa(n,s)}^n) - b_2^n(s,x_{\kappa(n,s)}^n))\,ds \\ \leq C\int_0^t \sup_{0\leq r\leq s}\mathbb{E}|e_{r\wedge\nu_{n,R}}^n|^2\,ds + Cn^{-\frac{5+\beta}{2}} + CR^{-\frac{2}{5}}n^{-2},\end{aligned}$$

where $\nu_{n,R}$ is the stopping time defined in (46).

Proof. First, for any $k = 1,\ldots,d$, applying Itô's formula to $b^{(k)}(x_t^n) - b^{(k)}(x_{\kappa(n,t)}^n)$ gives (41). Then, by (42) and (43), one obtains

$$\mathbb{E}\int_0^{t\wedge\nu_{n,R}} e_s^{n,(k)}(b^{(k)}(x_s^n) - b^{(k)}(x_{\kappa(n,s)}^n) - b_1^{n,(k)}(s,x_{\kappa(n,s)}^n) - b_2^{n,(k)}(s,x_{\kappa(n,s)}^n))\,ds \leq \sum_{i=1}^7 T_i(t) + T_8,$$

where

$$T_1(t) = \mathbb{E}\int_0^{t\wedge\nu_{n,R}} e_s^{n,(k)}\sum_{i=1}^d\int_{\kappa(n,s)}^s\left(\frac{\partial b^{(k)}(x_r^n)}{\partial x^{(i)}} - \frac{\partial b^{(k)}(x_{\kappa(n,r)}^n)}{\partial x^{(i)}}\right)b^{n,(i)}(x_{\kappa(n,r)}^n)\,dr\,ds,$$

$$T_2(t) = \mathbb{E}\int_0^{t\wedge\nu_{n,R}} e_s^{n,(k)}\sum_{i=1}^d\int_{\kappa(n,s)}^s\frac{\partial b^{(k)}(x_r^n)}{\partial x^{(i)}}(b_1^{n,(i)}(r,x_{\kappa(n,r)}^n) + b_2^{n,(i)}(r,x_{\kappa(n,r)}^n))\,dr\,ds,$$

$$T_3(t) = \mathbb{E}\int_0^{t\wedge\nu_{n,R}} e_s^{n,(k)}\sum_{i=1}^d\sum_{j=1}^m\int_{\kappa(n,s)}^s\left(\frac{\partial b^{(k)}(x_r^n)}{\partial x^{(i)}} - \frac{\partial b^{(k)}(x_{\kappa(n,r)}^n)}{\partial x^{(i)}}\right)\sigma^{n,(i,j)}(x_{\kappa(n,r)}^n)\,dw_r^j\,ds,$$

$$T_4(t) = \mathbb{E}\int_0^{t\wedge\nu_{n,R}} e_s^{n,(k)} \sum_{i=1}^{d}\sum_{j=1}^{m}\int_{\kappa(n,s)}^{s} \frac{\partial b^{(k)}(x_r^n)}{\partial x^{(i)}}\sigma_M^{n,(i,j)}(r, x_{\kappa(n,r)}^n)\, dw_r^j\, ds,$$

$$T_5(t) = \frac{1}{2}\mathbb{E}\int_0^{t\wedge\nu_{n,R}} e_s^{n,(k)} \sum_{i,l=1}^{d}\sum_{j=1}^{m}\int_{\kappa(n,s)}^{s} \left(\frac{\partial^2 b^{(k)}(x_r^n)}{\partial x^{(i)}\partial x^{(l)}} - \frac{\partial^2 b^{(k)}(x_{\kappa(n,r)}^n)}{\partial x^{(i)}\partial x^{(l)}}\right) \times \sigma^{n,(i,j)}(x_{\kappa(n,r)}^n)\sigma^{n,(l,j)}(x_{\kappa(n,r)}^n)\, dr\, ds,$$

$$T_6(t) = \frac{1}{2}\mathbb{E}\int_0^{t\wedge\nu_{n,R}} e_s^{n,(k)} \sum_{i,l=1}^{d}\sum_{j=1}^{m}\int_{\kappa(n,s)}^{s} \frac{\partial^2 b^{(k)}(x_r^n)}{\partial x^{(i)}\partial x^{(l)}}(\sigma^{n,(i,j)}(x_{\kappa(n,r)}^n)\sigma_M^{n,(l,j)}(r, x_{\kappa(n,r)}^n) + \sigma_M^{n,(i,j)}(r, x_{\kappa(n,r)}^n)\tilde{\sigma}^{n,(l,j)}(r, x_{\kappa(n,r)}^n))\, dr\, ds,$$

$$T_7(t) = C\int_0^t \sup_{0\le r\le s}\mathbb{E}|e_{r\wedge\nu_{n,R}}^n|^2\, ds,$$

$$T_8(t) = Cn^{-5}.$$

To estimate $T_1(t)$, one applies Young's inequality and Remark 1 to obtain

$$T_1(t) \le C\int_0^t \sup_{0\le r\le s}\mathbb{E}|e_{r\wedge\nu_{n,R}}^n|^2\, ds + Cn^{-1}\int_0^t\int_{\kappa(n,s)}^{s}\mathbb{E}(1+|x_r^n|+|x_{\kappa(n,r)}^n|)^{4\rho}|x_r^n - x_{\kappa(n,r)}^n|^2\, dr\, ds,$$

which by using Hölder's inequality implies

$$T_1(t) \le C\int_0^t \sup_{0\le r\le s}\mathbb{E}|e_{r\wedge\nu_{n,R}}^n|^2\, ds + Cn^{-1}\int_0^t\int_{\kappa(n,s)}^{s}\left(\mathbb{E}(1+|x_r^n|^{p_0}+|x_{\kappa(n,r)}^n|^{p_0})\right)^{\frac{4\rho}{p_0}}\left(\mathbb{E}|x_r^n - x_{\kappa(n,r)}^n|^{\frac{2p_0}{p_0-4\rho}}\right)^{\frac{p_0-4\rho}{p_0}} dr\, ds.$$

Thus, by Lemma 6 and Lemma 3, one obtains

$$T_1(t) \le C\int_0^t \sup_{0\le r\le s}\mathbb{E}|e_{r\wedge\nu_{n,R}}^n|^2\, ds + Cn^{-3},$$

for any $n\in\mathbb{N}$. For $T_2(t)$, $T_5(t)$ and $T_6(t)$, the same results can be obtained by the direct application of Cauchy-Schwarz inequality combining with previous Lemmas and Remarks. The rest of the proof will mainly focus on obtaining estimates for $T_3(t)$ and $T_4(t)$. For any $r\in[0,T]$, $i,k = 1,\ldots,d$ and $j = 1,\ldots,m$, denote by

$$\mathbb{T}_r^{(i,j,k)} := \left(\frac{\partial b^{(k)}(x_r^n)}{\partial x^{(i)}} - \frac{\partial b^{(k)}(x_{\kappa(n,r)}^n)}{\partial x^{(i)}}\right)\sigma^{n,(i,j)}(x_{\kappa(n,r)}^n) + \frac{\partial b^{(k)}(x_r^n)}{\partial x^{(i)}}\sigma_M^{n,(i,j)}(r, x_{\kappa(n,r)}^n).$$

Then, applying Remark 1 and Hölder's inequality yields

$$\begin{aligned}\mathbb{E}|\mathbb{T}_r^{(i,j,k)}|^p &= \mathbb{E}\left|\left(\frac{\partial b^{(k)}(x_r^n)}{\partial x^{(i)}} - \frac{\partial b^{(k)}(x_{\kappa(n,r)}^n)}{\partial x^{(i)}}\right)\sigma^{n,(i,j)}(x_{\kappa(n,r)}^n) + \frac{\partial b^{(k)}(x_r^n)}{\partial x^{(i)}}\sigma_M^{n,(i,j)}(r, x_{\kappa(n,r)}^n)\right|^p \\ &\le C\left(\mathbb{E}(1+|x_r^n|+|x_{\kappa(n,r)}^n|)^{p_0}\right)^{\frac{3\rho p}{2p_0}}\left(\mathbb{E}|x_r^n - x_{\kappa(n,r)}^n|^{\frac{2pp_0}{2p_0-3\rho p}}\right)^{\frac{2p_0-3\rho p}{2p_0}} \\ &\quad + C\left(\mathbb{E}(1+|x_r^n|)^{p_0}\right)^{\frac{\rho p}{p_0}}\left(\mathbb{E}|\sigma_M^{n,(i,j)}(r, x_{\kappa(n,r)}^n)|^{\frac{pp_0}{p_0-\rho p}}\right)^{\frac{p_0-\rho p}{p_0}},\end{aligned}$$

which, by using Lemma 6 and Lemma 5, implies

$$\sup_{r\le T}\mathbb{E}|\mathbb{T}_r^{(i,j,k)}|^p \le Cn^{-\frac{p}{2}}, \tag{47}$$

for $p\le \frac{2p_0}{7\rho+2}$. Due to (44) and (45) in the proof of Lemma 9, one can also obtain the following estimate

$$\mathbb{E}\left|\sum_{i=1}^d\sum_{j=1}^m\int_{\kappa(n,s)}^s \mathbb{T}_r^{(i,j,k)}\,dw_r^j\right|^2 \le 2\mathbb{E}|I_4(t)|^2+2\mathbb{E}|I_6(t)|^2\le Cn^{-2}. \tag{48}$$

Then, one writes

$$\begin{aligned}
T_3(t)+T_4(t):=&\,\mathbb{E}\int_0^{t\wedge\nu_{n,R}} e_s^{n,(k)}\sum_{i=1}^d\sum_{j=1}^m\int_{\kappa(n,s)}^s \mathbb{T}_r^{(i,j,k)}\,dw_r^j\,ds\\
=&\,\mathbb{E}\int_0^{t\wedge\nu_{n,R}} (e_s^{n,(k)}-e_{\kappa(n,s)}^{n,(k)})\sum_{i=1}^d\sum_{j=1}^m\int_{\kappa(n,s)}^s \mathbb{T}_r^{(i,j,k)}\,dw_r^j\,ds\\
&+\mathbb{E}\int_0^{t\wedge\nu_{n,R}} e_{\kappa(n,s)}^{n,(k)}\sum_{i=1}^d\sum_{j=1}^m\int_{\kappa(n,s)}^s \mathbb{T}_r^{(i,j,k)}\,dw_r^j\,ds.
\end{aligned}$$

Note that the second term above is not zero. However, by using Lemma 10, one obtains

$$\begin{aligned}
&\mathbb{E}\int_0^{t\wedge\nu_{n,R}} e_{\kappa(n,s)}^{n,(k)}\sum_{i=1}^d\sum_{j=1}^m\int_{\kappa(n,s)}^s \mathbb{T}_r^{(i,j,k)}\,dw_r^j\,ds\\
&=\mathbb{E}\int_0^{t} \mathbf{1}_{\{s\le\nu_{n,R}\}}e_{\kappa(n,s)\wedge\nu_{n,R}}^{n,(k)}\sum_{i=1}^d\sum_{j=1}^m\int_{\kappa(n,s)}^s \mathbb{T}_r^{(i,j,k)}\,dw_r^j\,ds\\
&=\mathbb{E}\int_0^{t} e_{\kappa(n,s)\wedge\nu_{n,R}}^{n,(k)}\sum_{i=1}^d\sum_{j=1}^m\int_{\kappa(n,s)}^s \mathbb{T}_r^{(i,j,k)}\,dw_r^j\,ds\\
&\qquad-\mathbb{E}\int_0^{t} \mathbf{1}_{\{s>\nu_{n,R}\}}e_{\kappa(n,s)\wedge\nu_{n,R}}^{n,(k)}\sum_{i=1}^d\sum_{j=1}^m\int_{\kappa(n,s)}^s \mathbb{T}_r^{(i,j,k)}\,dw_r^j\,ds,
\end{aligned}$$

where the first term is zero since $\kappa(n,s)\wedge\nu_{n,R}$ is $\mathcal{F}_{\kappa(n,s)}$-measurable. Then, applying Young's inequality, Hölder's inequality to the second term yield

$$\begin{aligned}
&\mathbb{E}\int_0^{t\wedge\nu_{n,R}} e_{\kappa(n,s)}^{n,(k)}\sum_{i=1}^d\sum_{j=1}^m\int_{\kappa(n,s)}^s \mathbb{T}_r^{(i,j,k)}\,dw_r^j\,ds\\
&\le C\int_0^t\sup_{0\le r\le s}\mathbb{E}|e_{r\wedge\nu_{n,R}}^n|^2\,ds + C\int_0^t(\mathbb{P}\left(s>\nu_{n,R}\right))^{\frac15}\left(\mathbb{E}\left|\sum_{i=1}^d\sum_{j=1}^m\int_{\kappa(n,s)}^s \mathbb{T}_r^{(i,j,k)}\,dw_r^j\right|^{\frac52}\right)^{\frac45}ds\\
&\le C\int_0^t\sup_{0\le r\le s}\mathbb{E}|e_{r\wedge\nu_{n,R}}^n|^2\,ds + CR^{-\frac25}n^{-2},
\end{aligned}$$

where the last inequality holds due to (47). One may notice that (47) holds only when $\frac52\le\frac{2p_0}{7\rho+2}$, which implies $p_0\ge\frac54(7\rho+2)$. However, as $\frac54(7\rho+2)\le 2(5\rho+1)$ for all $\rho\ge2$, by assuming $p_0 \geq 2(5\rho+1)$, (47) holds automatically for $p=\frac{5}{2}$. Furthermore, $T_3(t)+T_4(t)$ can be expressed as

$$
\begin{aligned}
T_3(t)+T_4(t) =& \mathbb{E}\int_0^{t\wedge\nu_{n,R}}\int_{\kappa(n,s)}^s \bar{b}^{n,(k)}(r,x^n_{\kappa(n,r)})\,dr\sum_{i=1}^d\sum_{j=1}^m\int_{\kappa(n,s)}^s \mathbb{T}_r^{(i,j,k)}\,dw_r^j\,ds \\
&+\mathbb{E}\int_0^{t\wedge\nu_{n,R}}\sum_{v=1}^m\int_{\kappa(n,s)}^s \bar{\sigma}^{n,(k,v)}(r,x^n_{\kappa(n,r)})\,dw_r^v\sum_{i=1}^d\sum_{j=1}^m\int_{\kappa(n,s)}^s \mathbb{T}_r^{(i,j,k)}\,dw_r^j\,ds \\
&+C\int_0^t\sup_{0\leq r\leq s}\mathbb{E}|e^n_{r\wedge\nu_{n,R}}|^2\,ds+CR^{-\frac{2}{5}}n^{-2},
\end{aligned}
$$

where $\bar{b}^{n,(k)}(t,x^n_{\kappa(n,t)})=b^{(k)}(x_t)-\tilde{b}^{n,(k)}(t,x^n_{\kappa(n,t)})$ and $\bar{\sigma}^{n,(k,v)}(t,x^n_{\kappa(n,t)})=\sigma^{(k,v)}(x_t)-\tilde{\sigma}^{n,(k,v)}(t,x^n_{\kappa(n,t)})$. One observes that $T_3(t)+T_4(t)$ can be expanded as

$$
\begin{aligned}
&T_3(t)+T_4(t)\\
&\quad=\mathbb{E}\int_0^{t\wedge\nu_{n,R}}\int_{\kappa(n,s)}^s (b^{(k)}(x_r)-b^{(k)}(x_r^n))\,dr\sum_{i=1}^d\sum_{j=1}^m\int_{\kappa(n,s)}^s \mathbb{T}_r^{(i,j,k)}\,dw_r^j\,ds \\
&\quad+\mathbb{E}\int_0^{t\wedge\nu_{n,R}}\int_{\kappa(n,s)}^s (b^{(k)}(x_r^n)-b^{(k)}(x^n_{\kappa(n,r)})-b_1^{n,(k)}(r,x^n_{\kappa(n,r)})-b_2^{n,(k)}(r,x^n_{\kappa(n,r)}))\,dr \\
&\qquad\qquad\times\sum_{i=1}^d\sum_{j=1}^m\int_{\kappa(n,s)}^s \mathbb{T}_r^{(i,j,k)}\,dw_r^j\,ds \\
&\quad+\mathbb{E}\int_0^{t\wedge\nu_{n,R}}\int_{\kappa(n,s)}^s (b^{(k)}(x^n_{\kappa(n,r)})-b^{n,(k)}(x^n_{\kappa(n,r)})\,dr\sum_{i=1}^d\sum_{j=1}^m\int_{\kappa(n,s)}^s \mathbb{T}_r^{(i,j,k)}\,dw_r^j\,ds \\
&\quad+\sum_{v=j=1}^m\sum_{i=1}^d\mathbb{E}\int_0^{t\wedge\nu_{n,R}}\int_{\kappa(n,s)}^s (\sigma^{(k,v)}(x_r)-\sigma^{(k,v)}(x_r^n))\mathbb{T}_r^{(i,j,k)}\,dr\,ds \\
&\quad+\sum_{v=j=1}^m\sum_{i=1}^d\mathbb{E}\int_0^{t\wedge\nu_{n,R}}\int_{\kappa(n,s)}^s (\sigma^{(k,v)}(x_r^n)-\sigma^{(k,v)}(x^n_{\kappa(n,r)})-\sigma_M^{n,(k,v)}(r,x^n_{\kappa(n,r)}))\mathbb{T}_r^{(i,j,k)}\,dr\,ds \\
&\quad+\sum_{v=j=1}^m\sum_{i=1}^d\mathbb{E}\int_0^{t\wedge\nu_{n,R}}\int_{\kappa(n,s)}^s (\sigma^{(k,v)}(x^n_{\kappa(n,r)})-\sigma^{n,(k,v)}(x^n_{\kappa(n,r)}))\mathbb{T}_r^{(i,j,k)}\,dr\,ds \\
&\quad+C\int_0^t\sup_{0\leq r\leq s}\mathbb{E}|e^n_{r\wedge\nu_{n,R}}|^2\,ds+CR^{-\frac{2}{5}}n^{-2},
\end{aligned}
$$

which implies due to Remark 1, Young's inequality and Cauchy-Schwarz inequality

$$
\begin{aligned}
&T_3(t)+T_4(t)\\
&\leq C\mathbb{E}\int_0^{t\wedge\nu_{n,R}}\left(\int_{\kappa(n,s)}^s(1+|x_r|+|x_r^n|)^{2\rho}\,dr\int_{\kappa(n,s)}^s|e_r^n|^2\,dr\right)^{\frac{1}{2}}\left|\sum_{i=1}^d\sum_{j=1}^m\int_{\kappa(n,s)}^s \mathbb{T}_r^{(i,j,k)}\,dw_r^j\right|\,ds \\
&+C\int_0^t\left(\mathbb{E}\left|\int_{\kappa(n,s)}^s|b(x_r^n)-b(x^n_{\kappa(n,r)})-b_1^n(r,x^n_{\kappa(n,r)})-b_2^n(r,x^n_{\kappa(n,r)})|\,dr\right|^2\right.\\
&\qquad\qquad\left.\times\sum_{i=1}^d\sum_{j=1}^m\mathbb{E}\left|\int_{\kappa(n,s)}^s \mathbb{T}_r^{(i,j,k)}\,dw_r^j\right|^2\right)^{1/2}\,ds
\end{aligned}
$$

$$
\begin{aligned}
&+ C\int_0^t n\times n^{-1}\mathbb{E}\int_{\kappa(n,s)}^s |b(x^n_{\kappa(n,r)}) - b^n(x^n_{\kappa(n,r)}|^2\,dr\,ds + Cn^{-1}\sum_{i=1}^d\sum_{j=1}^m\int_0^t \mathbb{E}\left|\int_{\kappa(n,s)}^s \mathbb{T}_r^{(i,j,k)}\,dw_r^j\right|^2 ds\\
&+ C\sum_{i=1}^d\sum_{j=1}^m\mathbb{E}\int_0^{t\wedge\nu_{n,R}}\int_{\kappa(n,s)}^s (1+|x_r|+|x_r^n|)^{\frac{\rho}{2}}|e_r^n||\mathbb{T}_r^{(i,j,k)}|\,dr\,ds\\
&+ C\sum_{i=1}^d\sum_{j=1}^m\int_0^t\int_{\kappa(n,s)}^s \sqrt{\mathbb{E}|\sigma(x_r^n) - \sigma(x^n_{\kappa(n,r)}) - \sigma^n_M(r, x^n_{\kappa(n,r)})|^2\mathbb{E}|\mathbb{T}_r^{(i,j,k)}|^2}\,dr\,ds\\
&+ C\sum_{i=1}^d\sum_{j=1}^m\int_0^t\int_{\kappa(n,s)}^s \sqrt{\mathbb{E}|\sigma(x^n_{\kappa(n,r)}) - \sigma^n(x^n_{\kappa(n,r)})|^2\mathbb{E}|\mathbb{T}_r^{(i,j,k)}|^2}\,dr\,ds\\
&+ C\int_0^t \sup_{0\le r\le s}\mathbb{E}|e^n_{r\wedge\nu_{n,R}}|^2\,ds + CR^{-\frac{2}{5}}n^{-2},
\end{aligned}
$$

for any $t\in[0,T]$. Then, by (48), (47), Lemma 9, Lemma 7, Lemma 8, Hölder's inequality and Cauchy-Schwarz inequality, one obtains

$$
\begin{aligned}
T_3(t)+T_4(t) \le{}& Cn^{-1}\mathbb{E}\int_0^t\int_{\kappa(n,s)}^s (1+|x_r|+|x_r^n|)^{2\rho}\,dr\left|\sum_{i=1}^d\sum_{j=1}^m\int_{\kappa(n,s)}^s \mathbb{T}_r^{(i,j,k)}\,dw_r^j\right|^2 ds\\
&+ Cn^{-1}\sum_{i=1}^d\sum_{j=1}^m\int_0^t\int_{\kappa(n,s)}^s \left(\mathbb{E}(1+|x_r|+|x_r^n|)^{p_0}\right)^{\frac{\rho}{p_0}}\left(\mathbb{E}|\mathbb{T}_r^{(i,j,k)}|^{\frac{2p_0}{p_0-\rho}}\right)^{\frac{p_0-\rho}{p_0}}\,dr\,ds\\
&+ C\mathbb{E}\int_0^t \sup_{0\le r\le s}\mathbb{E}|e^n_{r\wedge\nu_{n,R}}|^2\,ds + Cn^{-\frac{5+\beta}{2}} + Cn^{-3} + CR^{-\frac{2}{5}}n^{-2},
\end{aligned}
$$

which, by applying Hölder's inequality, yields

$$
\begin{aligned}
T_3(t)+T_4(t) \le{}& Cn^{-1}\sum_{i=1}^d\sum_{j=1}^m\int_0^t\left(n^{-\frac{p_0}{2\rho}+1}\mathbb{E}\int_{\kappa(n,s)}^s (1+|x_r|+|x_r^n|)^{p_0}\,dr\right)^{\frac{2\rho}{p_0}}\\
&\times\left(n^{-\frac{p_0}{p_0-2\rho}+1}\mathbb{E}\int_{\kappa(n,s)}^s |\mathbb{T}_r^{(i,j,k)}|^{\frac{2p_0}{p_0-2\rho}}\,dr\right)^{\frac{p_0-2\rho}{p_0}} ds\\
&+ C\mathbb{E}\int_0^t \sup_{0\le r\le s}\mathbb{E}|e^n_{r\wedge\nu_{n,R}}|^2\,ds + Cn^{-\frac{5+\beta}{2}} + CR^{-\frac{2}{5}}n^{-2},
\end{aligned}
$$

for any $t\in[0,T]$. Thus, by using Lemma 3 and (47), one obtains

$$
T_3(t)+T_4(t) \le C\int_0^t \sup_{0\le r\le s}\mathbb{E}|e^n_{r\wedge\nu_{n,R}}|^2\,ds + Cn^{-\frac{5+\beta}{2}} + CR^{-\frac{2}{5}}n^{-2},
$$

for any $n\in\mathbb{N}$. Finally, notice that

$$
\begin{aligned}
&\mathbb{E}\int_0^{t\wedge\nu_{n,R}} e_s^n(b(x_s^n) - b(x^n_{\kappa(n,s)}) - b_1^n(s,x^n_{\kappa(n,s)}) - b_2^n(s,x^n_{\kappa(n,s)}))\,ds\\
&\quad= \sum_{k=1}^d\mathbb{E}\int_0^{t\wedge\nu_{n,R}} e_s^{n,(k)}(b(x_s^{n,(k)}) - b(x^{n,(k)}_{\kappa(n,s)}) - b_1^n(s,x^{n,(k)}_{\kappa(n,s)}) - b_2^{n,(k)}(s,x^n_{\kappa(n,s)}))\,ds\\
&\quad\le C\int_0^t \sup_{0\le r\le s}\mathbb{E}|e^n_{r\wedge\nu_{n,R}}|^2\,ds + Cn^{-\frac{5+\beta}{2}} + CR^{-\frac{2}{5}}n^{-2},
\end{aligned}
$$

and the proof is complete. □

Proof of Theorem 1. Applying Itô's formula to $|e^n_{t\wedge\nu_{n,R}}|^2$ gives, almost surely,

$$|e^n_{t\wedge\nu_{n,R}}|^2 = 2\int_0^{t\wedge\nu_{n,R}} e^n_s \bar{b}^n(s,x^n_{\kappa(n,s)})\,ds + 2\int_0^{t\wedge\nu_{n,R}} e^n_s \bar{\sigma}^n(s,x^n_{\kappa(n,s)})\,dw_s + \int_0^{t\wedge\nu_{n,R}} |\bar{\sigma}^n(s,x^n_{\kappa(n,s)})|^2\,ds,$$

where $\nu_{n,R}$ is the stopping time defined in (46), $\bar{b}^n(t,x^n_{\kappa(n,t)}) = b(x_t) - \tilde{b}^n(t,x^n_{\kappa(n,t)})$ and $\bar{\sigma}^n(t,x^n_{\kappa(n,t)}) = \sigma(x_t) - \tilde{\sigma}^n(t,x^n_{\kappa(n,t)})$. Taking expectations on both sides and using Young's inequality yield, for any $\varepsilon > 0$,

$$\begin{aligned}
\mathbb{E}|e^n_{t\wedge\nu_{n,R}}|^2 \le 2\mathbb{E}&\int_0^{t\wedge\nu_{n,R}} e^n_s(b(x_s) - b(x^n_s))\,ds \\
&+ 2\mathbb{E}\int_0^{t\wedge\nu_{n,R}} e^n_s(b(x^n_s) - b(x^n_{\kappa(n,s)}) - b^n_1(s,x^n_{\kappa(n,s)}) - b^n_2(s,x^n_{\kappa(n,s)}))\,ds \\
&+ 2\mathbb{E}\int_0^{t\wedge\nu_{n,R}} e^n_s(b(x^n_{\kappa(n,s)}) - b^n(x^n_{\kappa(n,s)}))\,ds + (1+\varepsilon)\mathbb{E}\int_0^{t\wedge\nu_{n,R}} |\sigma(x_s) - \sigma(x^n_s)|^2\,ds \\
&+ C\mathbb{E}\int_0^{t\wedge\nu_{n,R}} |\sigma(x^n_s) - \sigma(x^n_{\kappa(n,s)}) - \sigma^n_M(s,x^n_{\kappa(n,s)})|^2\,ds \\
&+ C\mathbb{E}\int_0^{t\wedge\nu_{n,R}} |\sigma(x^n_{\kappa(n,s)}) - \sigma^n(x^n_{\kappa(n,s)})|^2\,ds.
\end{aligned}$$

for any $t \in [0,T]$. Then, by Cauchy-Schwarz inequality, one obtains

$$\begin{aligned}
\mathbb{E}|e^n_{t\wedge\nu_{n,R}}|^2 \le \mathbb{E}&\int_0^{t\wedge\nu_{n,R}} (2e^n_s(b(x_s) - b(x^n_s)) + (1+\varepsilon)|\sigma(x_s) - \sigma(x^n_s)|^2)\,ds \\
&+ 2\mathbb{E}\int_0^{t\wedge\nu_{n,R}} e^n_s(b(x^n_s) - b(x^n_{\kappa(n,s)}) - b^n_1(s,x^n_{\kappa(n,s)}) - b^n_2(s,x^n_{\kappa(n,s)}))\,ds \\
&+ \mathbb{E}\int_0^{t\wedge\nu_{n,R}} |b(x^n_{\kappa(n,s)}) - b^n(x^n_{\kappa(n,s)})|^2\,ds \\
&+ C\mathbb{E}\int_0^{t\wedge\nu_{n,R}} |\sigma(x^n_s) - \sigma(x^n_{\kappa(n,s)}) - \sigma^n_M(s,x^n_{\kappa(n,s)})|^2\,ds \\
&+ C\mathbb{E}\int_0^{t\wedge\nu_{n,R}} |\sigma(x^n_{\kappa(n,s)}) - \sigma^n(x^n_{\kappa(n,s)})|^2\,ds + \int_0^t \sup_{0\le r\le s} \mathbb{E}|e^n_{r\wedge\nu_{n,R}}|^2\,ds.
\end{aligned}$$

Since $p_1 > 2$, applying **A-3** to the first term, and applying Lemma 11, 7 and 8 yield

$$\sup_{0\le s\le t} \mathbb{E}|e^n_{s\wedge\nu_{n,R}}|^2 \le C\int_0^t \sup_{0\le r\le s} \mathbb{E}|e^n_{r\wedge\nu_{n,R}}|^2\,ds + Cn^{-(2+\beta)} + CR^{-\frac{2}{5}}n^{-2} < \infty,$$

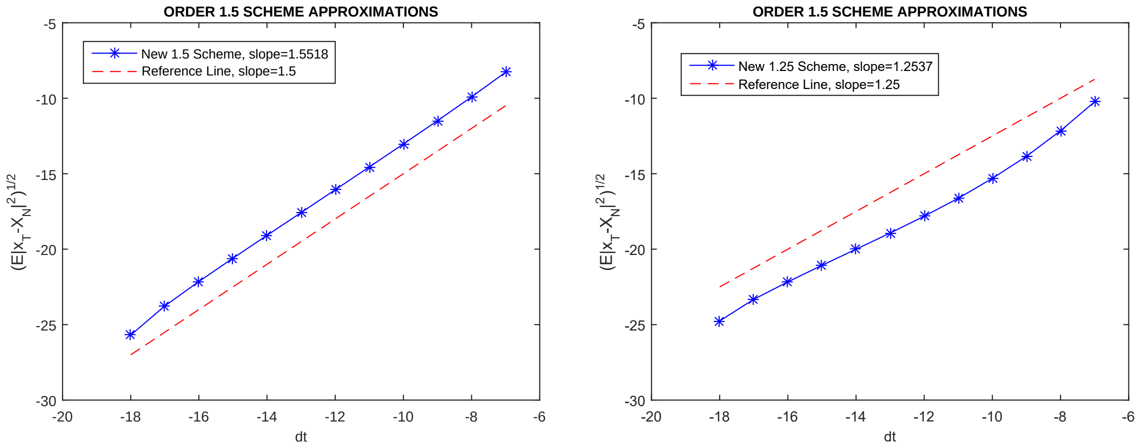
for any $t \in [0,T]$ and $n \in \mathbb{N}$. Finally, one applies Gronwall's lemma to obtain

$$\sup_{0\le s\le t} \mathbb{E}|e^n_{s\wedge\nu_{n,R}}|^2 \le Cn^{-(2+\beta)} + CR^{-\frac{2}{5}}n^{-2},$$

and the proof is complete by using Fatou's lemma, since the last term in the above inequality vanishes as R tends to infinity.

5 Simulation results

In this section, simulation results are provided to support the theoretical results in the previous sections. Consider $T = 1$, the step size $\Delta = t_{k+1} - t_k = 1/N$ for $N \in \mathbb{N}$, $t_0 = 0$, and



(a) $\beta = 1$

(b) $\beta = 0.5$

Figure 1: Rate of convergence of the new order 1.5 scheme with parameters $x_0 = 3$, $\xi = 0.02$ and $T = 1$. Denote by x_T and X_N respectively the true solution and the numerical approximation of the corresponding SDE at time T. The dashed red lines are the reference lines, and the blue doted lines are the numerical results obtained using the scheme.

$k \in \{0, \ldots, N-1\}$. For the case $d = m = 1$, the discrete version of the order 1.5 scheme (22) is as follows:

$$\begin{aligned} X_{k+1} = X_k &+ b^n \Delta + \sigma^n \Delta W + L^{n,1} b \Delta Z + \frac{1}{2} L^{n,0} b \Delta^2 \\ &+ \frac{1}{2} L^{n,1} \sigma ((\Delta W)^2 - \Delta) + L^{n,0} \sigma (\Delta W \Delta - \Delta Z) \\ &+ \frac{1}{2} L^{n,1} L^1 \sigma \left(\frac{1}{3} (\Delta W)^2 - \Delta \right) \Delta W, \end{aligned}$$

where the following conventions are used: $X_k = X_{t_k}$, $\Delta W = W_{t_{k+1}} - W_{t_k}$ and $\Delta Z = \int_{t_k}^{t_{k+1}} \int_{t_k}^{s} dW_r \, ds$. Note that ΔZ is normally distributed with mean zero, variance $\frac{1}{3}\Delta^3$, and covariance

$$\mathbb{E}(\Delta Z \Delta W) = \frac{1}{2} \Delta^2.$$

Then, the following two examples are considered. For the first example, the one-dimensional SDE is given by

$$dx_t = x_t(1 - x_t^2)dt + \xi(1 - x_t^2)dw_t, \quad \forall t \in [0, T], \tag{51}$$

where $T \geq 0$ and $\xi \in [-0.3086, 0.3086]$. As for the second example, one consider the SDE

$$dx_t = x_t(1 - |x_t|^3)dt + \xi |x_t|^{\frac{5}{2}} dw_t, \quad \forall t \in [0, T], \tag{52}$$

where $T \geq 0$ and $\xi \in [-0.2209, 0.2209]$. One can check (see Appendix) that the first example (51) satisfies the assumptions **A-1** to **A-5** with $\rho = 2$, whereas the second example (52) satisfies the assumptions with $\rho = 4$. As for the numerical results, Figure 1 above shows the rate of convergence of the scheme, and the approximations are obtained by simulating 1000 paths. Furthermore, Figure 1(a) illustrates that, for the case $\beta = 1$, the new explicit order 1.5 scheme has a rate of convergence estimate close to the theoretical result 1.5, which is 1.5518. Similarly, as shown in Figure 1(b), the slope of the blue line is equal to 1.2537, which supports the theoretical prediction 1.25. Note that the examples considered in this section are one dimensional. However, in order to implement such algorithms to real world problems where $d \geq 2$, the diffusion coefficient needs to satisfy the commutative condition. Otherwise, one needs to handle the associated Levy areas. One possible approach is to use a coupling technique (see [3]).

6 Appendix

1. Consider the one-dimensional SDE
$$dx_t = x_t(1 - x_t^2)dt + \xi(1 - x_t^2)dw_t, \quad \forall t \in [0, T].$$
   (a) **A-1** is satisfied as x_0 is taken to be a constant (i.e. $x_0 = 3$).

   (b) To verify **A-2**, one calculates
$$\begin{aligned} 2xb(x) + (p_0 - 1)|\sigma(x)|^2 &= 2x^2 - 2x^4 + (p_0 - 1)\xi^2(1 - x^2)^2 \\ &= (p_0 - 1)\xi^2 + 2(1 - \xi^2(p_0 - 1))x^2 + (\xi^2(p_0 - 1) - 2)x^4. \end{aligned}$$
   We require $\xi^2(p_0 - 1) - 2 \leq 0$, which implies $p_0 \leq \frac{2}{\xi^2} + 1$.

   (c) As for **A-3**, one writes
$$\begin{aligned} &2(x - \bar{x})(b(x) - b(\bar{x})) + (p_1 - 1)|\sigma(x) - \sigma(\bar{x})|^2 \\ &\quad = 2(x - \bar{x})((x - x^3) - (\bar{x} - \bar{x}^3)) + (p_1 - 1)\xi^2|(1 - x^2) - (1 - \bar{x}^2)|^2 \\ &\quad = 2(x - \bar{x})^2 - 2(x - \bar{x})^2((x + \bar{x})^2 - x\bar{x}) + (p_1 - 1)\xi^2|x + \bar{x}|^2|x - \bar{x}|^2 \\ &\quad \leq 2(x - \bar{x})^2 + (x - \bar{x})^2 \left((p_1 - 1)\xi^2|x + \bar{x}|^2 - (x + \bar{x})^2\right). \end{aligned}$$
   Then, in order to guarantee $2(x-\bar{x})(b(x)-b(\bar{x}))+(p_1-1)|\sigma(x)-\sigma(\bar{x})|^2 \leq K|x-\bar{x}|^2$ is satisifed for some $K > 0$, we require $p_1 \in (2, \frac{1}{\xi^2} + 1]$.

   (d) The second derivative of $b(x) = x(1 - x^2)$ is $-6x$, then **A-4** is satisfied with $\rho \geq 2$ since
$$\left| \frac{\partial^2 b(x)}{\partial x^2} - \frac{\partial^2 b(\bar{x})}{\partial \bar{x}^2} \right| \leq 6|x - \bar{x}|$$
   (e) Similary, one can calculate the second derivative of $\sigma(x) = \xi(1 - x^2)$, which is -2ξ. The assumption **A-5** is satisfied with $\rho \geq 2$.

   We choose ρ to be 2, then, since it is assumed in Theorem 1 that $p_0 \geq 2(5\rho + 1) = 22$, one obtains $\xi \in [-0.3086, 0.3086]$ by using $p_0 \in [22, \frac{2}{\xi^2} + 1]$ and $p_1 \in (2, \frac{1}{\xi^2} + 1]$.

2. As for the second example, consider the one-dimensional SDE
$$dx_t = x_t(1 - |x_t|^3)dt + \xi|x_t|^{\frac{5}{2}}dw_t, \quad \forall t \in [0, T].$$
   (a) We take $x_0 = 3$, therefore **A-1** is satisfied.

   (b) As for **A-2**, one calculates
$$\begin{aligned} 2xb(x) + (p_0 - 1)|\sigma(x)|^2 &= 2x^2 - 2|x|^5 + (p_0 - 1)\xi^2|x|^5 \\ &= 2x^2 + ((p_0 - 1)\xi^2 - 2)|x|^5. \end{aligned}$$
   To guarantee **A-2** is satisfied, we require $p_0 \leq \frac{2}{\xi^2} + 1$.

(c) To verify **A-3**, one calculates the following

$$
\begin{aligned}
&2(x-\bar{x})(b(x)-b(\bar{x}))+(p_1-1)|\sigma(x)-\sigma(\bar{x})|^2\\
&\quad=2(x-\bar{x})((x-x|x|^3)-(\bar{x}-\bar{x}|\bar{x}|^3))+(p_1-1)\xi^2\left||x|^{\frac{5}{2}}-|\bar{x}|^{\frac{5}{2}}\right|^2\\
&\quad=2(x-\bar{x})^2-2(|x|^5-x\bar{x}|x|^3-x\bar{x}|\bar{x}|^3+|\bar{x}|^5)+(p_1-1)\xi^2\left||x|^{\frac{5}{2}}-|\bar{x}|^{\frac{5}{2}}\right|^2\\
&\quad\le 2(x-\bar{x})^2+\left(-2|x|^5-2|\bar{x}|^5+\frac{6}{5}|x|^5+\frac{6}{5}|\bar{x}|^5+\frac{8}{5}|x|^{\frac{5}{2}}|\bar{x}|^{\frac{5}{2}}\right)+(p_1-1)\xi^2\left||x|^{\frac{5}{2}}-|\bar{x}|^{\frac{5}{2}}\right|^2\\
&\quad=2(x-\bar{x})^2+\left((p_1-1)\xi^2-\frac{4}{5}\right)\left||x|^{\frac{5}{2}}-|\bar{x}|^{\frac{5}{2}}\right|^2.
\end{aligned}
$$

Therefore, we require $p_1\in(2,\frac{4}{5\xi^2}+1]$ for **A-3** to be satisfied.

(d) The second derivative of $b(x)=x(1-|x|^3)$ is $-12x|x|$, then **A-4** is satisfied with $\rho\ge 3$ since

$$
\begin{aligned}
\left|\frac{\partial^2 b(x)}{\partial x^2}-\frac{\partial^2 b(\bar{x})}{\partial \bar{x}^2}\right| &\le 12|\bar{x}|\bar{x}|-x|x||\\
&=12|\bar{x}|\bar{x}|-x|\bar{x}|+x|\bar{x}|-x|x||\\
&\le 12|\bar{x}||\bar{x}-x|+|x||\bar{x}-x|\\
&\le 12(|x|+|\bar{x}|)|\bar{x}-x|\\
&\le 12(1+|x|+|\bar{x}|)|\bar{x}-x|.
\end{aligned}
$$

(e) The second derivative of $\sigma(x)=\xi|x|^{\frac{5}{2}}$ is $\frac{15}{4}\xi|x|^{\frac{1}{2}}$, then one obtains

$$
\left|\frac{\partial^2\sigma(x)}{\partial x^2}-\frac{\partial^2\sigma(\bar{x})}{\partial\bar{x}^2}\right|\le\frac{15}{4}|\xi|\left||x|^{\frac{1}{2}}-|\bar{x}|^{\frac{1}{2}}\right|\le\frac{15}{4}|\xi||x-\bar{x}|^{\frac{1}{2}},
$$

which implies that **A-5** is satisfied with $\rho\ge 4$, and the last inequality holds since

$$
\left||x|^{\frac{1}{2}}-|\bar{x}|^{\frac{1}{2}}\right|^2\le\left||x|^{\frac{1}{2}}-|\bar{x}|^{\frac{1}{2}}\right|\left||x|^{\frac{1}{2}}+|\bar{x}|^{\frac{1}{2}}\right|\le||x|-|\bar{x}||\le|x-\bar{x}|.
$$

We choose $\rho=4$, then, as it is assumed in Theorem 1 that $p_0\ge 2(5\rho+1)=42$, one obtains $\xi\in[-0.2209,0.2209]$ by using $p_0\in[42,\frac{2}{\xi^2}+1]$ and $p_1\in(2,\frac{4}{5\xi^2}+1]$.

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